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A TREATISE ON

REINFORCED CONCRETE

A TREATISE ON REINFORCED CONCRETE

BY

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INCLUDING THE NEW STANDARD NOTATION OF THE CONCRETE INSTITUTE

WITH A FOREWORD
ON
STANDARD NOTATION FOR ENGINEERING FORMULÆ
BY

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PREFACE

In the present volume the Author has endeavoured to set forth as clearly as possible the general characteristics and distinctive properties of reinforced concrete and its constituents, to discuss in a systematic manner the principles underlying the design of homogeneous members, and to show how these principles may be applied to the evolution of formulæ for the design of reinforced concrete members of different classes.

It is scarcely necessary to give in this Preface a synopsis of the work, as that will be found in the table of contents on another page, and the method of treatment will be readily gathered by glancing through the pages of the book itself. The subject of reinforced concrete cannot be dealt with satisfactorily in all its ramifications in a single volume of moderate size. Therefore the Author has restricted his efforts to what he hopes may be considered a thorough exposition of fundamental principles, and to the presentation of a complete series of formulæ for the principal classes of members employed in engineering and building construction.

The Author desires to express his great indebtedness to Mr. E. Fiander Etchells for many valuable hints and assistance in the correction of proofs, although he does not suggest that all the views expressed in the book are shared by that gentleman. He is further indebted to Mr. Etchells for the contribution of the Foreword on Standard Notation for Engineering Formulæ. The Author has also to thank the Concrete Institute for placing at his disposal the new Standard Notation approved in April, 1920.

This is the first book in the English language embodying the improved notation, which almost automatically imprints itself upon the memory of the user, and has been found by the Author to be convenient and satisfactory in every way.

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STANDARD NOTATION

FOREWORD

ON

STANDARD NOTATION FOR ENGINEERING FORMULÆ

BY E. FIANDER ETCHELLS.

DESIDERATA

THE primary conventions of the ordinary mathematical notation were established before Caxton introduced the printing press which bears his name.

So long as equations were printed from loose type set up by hand from many different founts it was generally possible to imitate, more or less closely, the forms given in the manuscript original.

The setting up of mathematical equations was, however, always regarded as special and difficult work for the compositor; and the general introduction of type-setting machines, such as the Linotype and Monotype, has made these difficulties more generally apparent.

Sometimes the ordinary context is set up by machines, and the interposed equations are set up by hand in small separate metal frames or "sticks."

In some instances the context may be set up by machines in London, and the equations may be set up by hand by a compositor in Cambridge or elsewhere.

Occasionally the setting up of equations from loose type is given up altogether on account of the trouble and expense, and either a lithographic or an engraving process is adopted instead.

Lithographic and engraving processes are, however, only suitable for complete sheets filled with equations and would be impracticable for the various mathematical expressions which occur incidentally and promiscuously in the pages of technical books.

Some authors employ Old Gothic and other highly decorated letters, but although these can be typed and printed, yet they are too complex for rapid work at the blackboard.

It is therefore desirable to make such adjustments in the current mathematical notation as will enable mathematical expressions to be—

(1) written in chalk on the blackboard;

(2) written on the typewriter;

(3) set up in type by machinery

in the same straightforward manner as an ordinary passage from a newspaper, and the mathematical expression when written or printed should be capable of being read and interpreted as easily as any other piece of simple shorthand.

The Standard Notation in which the writer has been interested for the last twenty-five years recognizes that—

(1) The mathematical conventions of the Pre-Caxton period are not always suitable for the Post-Linotype period.

(2) No new convention must be introduced if an existing or classical convention can be found and strengthened.

(3) The notation must not be less logical, but more logical than the notation found in the existing text-books.

(4) The symbols for the pure numbers of Abstract Mathematics may be represented by the letters of the alphabet arranged in order as at present, but the magnitudes referred to in Applied Mathematics should be represented by the initial letters which remain after a process of elimination in accordance with a few easily remembered rules.

ORIGIN OF THE CONFLICTING CONVENTIONS

More than a thousand years ago, India, Arabia, Greece, Italy and Spain each contributed their quota to the science of mathematics, and several divergent systems of notation came into use in the different countries, and at different times.

According to one system the juxtaposition of two symbols indicated addition. This system survives in our present method of setting out numbers, where 23 means 20 plus 3, and not 2 multiplied by 3.

This principle is seen more clearly still in the Roman numerals. For example, XXIII means $10 + 10 + 1 + 1 + 1 = 20 + 3$.

According to a second system, the juxtaposition of two symbols indicated multiplication. This system survives in our ordinary algebra where ab is used to indicate the product of a multiplied by b.

According to a third system the juxtaposition of two symbols indicated some geometrical relationship. This system survives in our ordinary geometrical diagrams in which AB denotes the terminal points of a line, and not A multiplied by B.

According to a fourth system, the juxtaposition of two letters denotes the abbreviation of two words until nothing remains except the two initial letters. This system survives when a binary symbol such as dt is used to denote the "differential of the time," and not d multiplied by t.

The continued usage of these four divergent systems side by side has led to the existence in modern text books of innumerable ambiguities and inconsistencies. Thousands upon thousands of examples could be quoted if it were profitable to do so.

USES OF THE POINT

In the new standard notation which seeks to eliminate some of the foregoing inconsistencies, three points are used, viz.—

(1) The *high* point, which is used as the decimal separatrix. Example, 7·5 = seven and five tenths.

(2) The *middle* point which is used as the sign of multiplication between two letters which represent the two factors. Example, $a \cdot b$ = a multiplied by b.
(3) The *low* point which is used as the sign of abbreviation. Example, B.H.P. = Brake Horse Power, etc.

We now propose to add a few remarks in respect of each of these points.

(1)—The Decimal Separatrix

As mathematics is pre-eminently cosmopolitan in its character, it is desirable that its symbols of operation and its ideographic symbols should also be cosmopolitan and free from ambiguity. Indubitably the best symbol for the decimal separatrix is the high point.

The low point which is used by the Americans, often denotes multiplication in England. Thus 3.4 would mean three plus four tenths in America, but in England it would mean three multiplied by four. (*Vide A New Algebra*, by Barnard & Child, 1913 Edition, page 9.) In France, 8.001 would mean 8,000 plus 1. In America, 8.001 would mean 8 plus 1/1000.

The comma, which is used in France, Italy, and Germany as the decimal separatrix, is used in America and in England to divide the groups of thousands. Undoubtedly the high point is the sign most free from ambiguity and therefore the most suitable for international use. It is rapidly coming into general use among British printers, and it is hoped that other nations will adopt it too. It is given as the Standard English decimal point in the classic treatise by Legros and Grant on *Typographical Printing Surfaces*. It is not exactly novel in this country, for we notice that it was used twenty years ago in *Structural Iron and Steel*, by W. Noble Twelvetrees, and we have traced it in other books earlier than this.

(2)—The Signs of Multiplication

The middle point, which is used as the sign of multiplication between two literal symbols, such as $a \cdot b$ or $m \cdot n$ should never be omitted.

The sign of multiplication between two numerals is an $\times$ and this sign is never used between two letters.

Thus we may say $w = x \cdot y$, but not $w = x \times y$.

The sign of multiplication between a numeral and a literal symbol is simply the old device of juxtaposition so that $4Q$ means four times the *quantity* which is denoted by Q.

The unfailing use of the middle point as a sign of multiplication between literal symbols has very great and far-reaching advantages, for example: it obviates the use of subscript letters; eliminates the use of extra founts of type; *puts less strain on the eyesight by avoiding the use of the small type which is commonly used for subscripts;* and it enables binary or compound symbols to be set upon Linotype and Monotype machines without difficulty. It also cheapens the cost of producing technical books and involves no loss of clarity, and gives rise to no ambiguity.

Subscript letters were introduced by Leibnitz (1646–1716). They have always been a great nuisance to the printer and yet without them a suggestive notation seemed impossible.

There is nothing really revolutionary in the use of the middle point as a sign of multiplication for it is in common use in American Engineering text-books. We find it is in use in some German books. See Aachener Hütten-Actien Verein *Profil Album*, 1906, for example, where we find the equation

$$Ty = \frac{1}{12}\, b \cdot h^3$$

in which Ty is a binary symbol for one quantity, and where y is written on the line of printing and not as a subscript.

As a matter of fact, the point, as a sign of multiplication, goes back to a remote antiquity. Bhaskara, a Hindoo mathematician, who was born in 1114 A.D., sometimes separated the two factors of a product by means of a point; but he does not appear to have been very particular as to the exact position of the point; and it may have been that the point which began its life history as a sign of abbreviation, gradually acquired a new meaning as a sign of multiplication.

(3)—The Sign of Abbreviation

According to Husband, in very ancient Phoenician inscriptions there was no space between the separate words, but on the celebrated Moabite Stone ascribed to the ninth century B.C., there is a clear separation of words by single dots, and of sentences by vertical strokes. At a later date Latin inscriptions are found with the words separated by a single dot. At a later date still it is found that the Latin inscriptions which deal with the ordinary facts and designations of life, are generally expressed in conventionally abbreviated forms. The dot of word separation then becomes merged into or identical with the dot of abbreviation.

This principle survives in the Standard Notation in such forms as—

H.C.F. = Highest Common Factor.
O.M. = Overturning Moment.
S.M. = Stability Moment, etc.

This low point is, however, always omitted from equations, although it is retained in the context in accordance with usual custom. Thus, in equations we have only to deal with the high point and the middle point, and there can be no confusion with any points of punctuation or signs of abbreviation.

DERIVATION OF SYMBOLS

There are four distinct stages in the derivation of a mnemonic literal symbol. In the first stage we have the complete phrase which describes or defines the given magnitude.

This phrase is called the complete symbol or the "phrasic" symbol.

In the second stage the phrasic symbol is curtailed until the only residue is the dominant or distinguishing word or words of the phrase. The dominant or distinguishing words are called the "verbal" symbol.

In the third stage the verbal symbol is curtailed until the only residue is the initial syllable or syllables. The initial syllables are called the "syllabic" symbols.

In the fourth stage the syllabic symbols are curtailed until the only residue is the initial letter or letters.

ORDER OF QUANTITIES

The quantities in an expression should be sorted out into different kinds and arranged in the following order—

(1) Constants.
(2) Forces.
(3) Spaces.
(4) Times.
(5) Trigonometrical and other Functions.

This order happens to be alphabetical if the English terms are used, but the order is really an international tradition or convention, and it would not be altered if these quantities were described by other terms, in English or in any other language.

Following this order in greater detail, we would obtain the following results. Quantities of higher orders would be placed first, so that if we had spatial quantities to deal with they would fall into the following order—

(1) Volumes.
(2) Areas.
(3) Lengths.

If we had a series of lengths they would be placed in alphabetical order, in the national language employed in the context, for example: in English we have—

Example I.

(1) altitude.
(2) breadth.

Example II.

(1) breadth.
(2) depth.

Example III.

(1) breadth.
(2) height.

Example IV.

Let B = *Bending* moment on a certain beam.
W = *Weight* to be carried.
l = Effective *length*.
$B = W \cdot l/8$

Note that weight is a force, while length is a spatial dimension, and so we have the form $W \cdot l$ and not $l \cdot W$.

For the same reason we should speak of tons-feet or pounds-inches, rather than to write the quantities in one order, and then speak of them in the reverse order as is so often done.

Example V.

Let P = total *pressure* in pounds on each lineal foot of a retaining wall (not surcharged).
h = *height* of the wall in feet.
θ = angle of repose of the material measured from the horizontal plane.
w = *weight* of the material retained in pounds per cubic foot.

$$P = \frac{w \cdot h^2}{2} \cdot \frac{1 - \sin\theta}{1 + \sin\theta}$$

Note that these quantities are arranged in the order

(1) Force.
(2) Spatial dimensions.
(3) Trigonometrical expressions.

RELATIVE GREATERNESS

The standard notation takes advantage of the differences between the lower case letters and the capitals. The *lesser* letters (a, b, c, d, etc.), indicate relative lesserness, such as lineal dimensions, and forces on one unit of area, whilst the *greater* letters (A, B, C, D, etc.) indicate relative greaterness, such as forces on the total area, and *products* of lineal dimensions, such as moments, areas and volumes.

LOWER CASE LETTERS REPRESENTING ORDINAL NUMBERS

At one stage of Greek history the first 999 numerals were represented by the Greek "lower case" letters α, β, γ, etc., together with three strange letters from some earlier or alien alphabet. Twenty-seven letters were thus available.

The first group of nine letters represented the digits 1 to 9.

The second group of nine letters represented the tens from 10 to 90.

The third group of nine letters represented the hundreds from 100 to 900.

After the Greek system of numerals, there arose the Roman system, but this in its turn gave way to the Hindu-Arabic system.

After the lapse of more than 2,000 years, the introduction of the typewriter and the Linotype composing machine has led to a partial return to the Greek system of using letters to represent specific and particular numbers.

Thus, suppose we have to deal with a series of quantities, a first quantity, a second quantity, a third quantity and so on. These could be represented by Q_1, Q_2, Q_3, etc., but these inferior or subscript numerals would require a special fount of type on the Linotype. And on the typewriter it would be necessary for the typist to pause and turn the roller up for half a space, type the numeral, pause again, turn the roller down half a space, and then resume typing on the line of the context. All this is a waste of time and money.

A better method would be to represent these quantities by *Qa*, *Qb*, *Qc*, etc., which could be written on the typewriter with facility, and can be set up on the Linotype without the use of a set of inferior or subscript numerals.

The simpler method is therefore adopted in the Standard Notation, and it is rendered possible by the use of the middle point as a sign of multiplication.

Thus the product of the quantities would be represented by

$$Qa \cdot Qb \cdot Qc$$

TYPEWRITER KEYBOARDS

There is no difficulty in adding the high point and the middle point to the typewriter keyboard, in place of two of the rarely used symbols, which are to be found at the extreme right hand of the ordinary commercial machines.

As a matter of fact the Remington Typewriter Company supply a special Mathematical keyboard, which was designed originally for the Concrete Institute by the present writer. It embodies all the letters of the Standard keyboard in their standard order, but it also contains some additional mathematical signs, including the high point for the decimal separatrix and the middle point for the sign of multiplication between literal symbols.

The order of the new signs was arranged in collaboration with the Remington Company in such a manner as to reduce the cost of the change to a minimum by using pairs of characters already standardized by that company.

THE SUPERIOR DASH

The superior dash, *i.e.*, a dash above and to the right of a letter is a mnemonic device to indicate some dimension which is above, or to the right of some other corresponding dimension. Thus, if a is an *arm* measured below some axis, then a' is a corresponding arm measured above the same axis.

If b is a *breadth* measured to the left of some axis, then b' would represent a breadth to the right of that axis.

The minute mark can be used as a superior dash.

In typewritten work the sign which is used after a numeral to indicate that the dimension is in feet could be used after a literal symbol for the purpose described above.

STANDARD RATIOS

In the various branches of Engineering it is frequently possible to use some particular dimension as a unit of measurement. For example, the depth of a beam may be treated as a unit: then the length and breadth of the beam

and all its other lineal dimensions may be expressed in terms of the effective depth.

Suppose these lineal dimensions were represented by a, b, d, e, l, n, etc., where d represented the effective depth, it might be convenient to use the ratios a/d, b/d, e/d, l/d, n/d, etc.

Notice that in these ratios, the antecedents are a, b, e, l, n, etc., but the consequent in every case is d.

In the standard notation, where we have a series of ratios having a consequent which is common to them all, we use a special form of dash, so that

$$\begin{aligned} a_{\backslash} &= a/d \\ b_{\backslash} &= b/d \\ e_{\backslash} &= e/d \\ l_{\backslash} &= l/d \\ n_{\backslash} &= n/d, \text{ etc.} \end{aligned}$$

The practical advantages of this device are best discovered by actual trial.

The general idea of a series of standard ratios is very old, but the credit (or the blame) for the introduction of a sloping dash on or through the line to denote these standard ratios would appear to lie with the present writer.

The contra-italic slope of the dash was suggested by Mr. L. A. Legros, M.Inst.C.E., one of the greatest living authorities on typographical printing surfaces.

The advantage of the contra-italic slope is that it eliminates the risk of confusion with the comma or the inferior number one. The contra-italic dash was adopted by the Concrete Institute on the 14th April, 1920, after Mr. Twelvetrees' book was completed, and the sub-dash employed in his pages must be regarded as having identically the same significance as the contra-italic dash referred to above.

OTHER RULES

There are other rules which lead up to the standardization of the symbols and their arrangement in a definite order in

formulae. These rules will be found in the Report on Standard Notation for Engineering Formulae, by the Science Committee of the Concrete Institute (London).

In this foreword we have dealt with certain additional features and symbols, which have been adopted by the Concrete Institute, subsequent to the publication of the Report referred to above.

At this stage we should like to record our indebtedness to Prof. Adams, M.Inst.C.E., who for more than twelve years has approved of the efforts of the present writer to find a practical means of abolishing the objectionable subscript letters with the least disturbance of the medieval conventions of pure mathematics.

It would, however, be too long a task to particularize on all these points, for many critical friends and friendly critics have, in one way or another, contributed their quota to that system of mathematical shorthand which is known as "The Standard Notation."

CONCLUSION

In conclusion, it is of interest to note that this book, by W. Noble Twelvetrees, which in its earlier pages quotes some of the historical fundamental equations in their original accidental notation is, at the same time, the first book in the English language which contains a mnemonic notation without the use of subscripts. This, however, is not the book's only claim to distinction, for it probably contains a more complete set of formulæ for the resistance moments of various types of reinforced concrete beams than any book hitherto published in this country.

STANDARD NOTATION

OF

THE CONCRETE INSTITUTE

For calculations in respect of Reinforced Concrete and other branches of Structural Engineering, including the symbols approved on the 14th April, 1920.

Applicability to Various Systems of Algebra.

The Standard Notation is applicable to—

(1) the pure algebra of abstract numbers;

(2) the ordinary applied algebra of cardinal numbers;

(3) magnitudic algebra; *i.e.*, the algebra of magnitudes, and not of numbers only.(1)

Note as to Units.

The Standard Notation is independent of the system of units employed, but the description of the unit should be added to the definition of each symbol.

The list of symbols must be considered as being incomplete until the user has specified the units proposed for adoption in every case. Except where otherwise stated, each symbol represents a real magnitude, *i.e.*, the product (numeric × unit).

This method affords great help in working out engineering formulæ, and is of great assistance to those who have to check the numerical results.

(1) A work on Magnitudic Algebra is in course of preparation.

PART I

ABBREVIATIONS

In general	A	= *area*, total *area*.
In general	A_r	= *areal* ratio = a ratio between two *areas* as specified by the user.
In general	Aa, Ab = Ac, etc. =	a series of *areas* which have no specific names, but are indicated on some diagram.
In R.C. beams	A	= cross sectional *area* of tension reinforcement.
In R.C. struts	A	= effective *area* which must be fully and clearly defined by the user.
In R.C. pillars	A	= effective *area* which must be fully and clearly defined by the user.
In general	Ab	= cross sectional *area* of one *bar*.
In R.C. beams	Ab	= cross sectional *area* of *beam* which must be fully and clearly defined by the user.
In R.C. pillars	Ab	= cross sectional *area* of one *bar* of the *binding*.
In R.C. beams	Ac	= cross sectional *area* of *compression* reinforcement.
In R.C. beams	Ad	= cross sectional *area* of *diagonal* web members in a specified length of a beam which must be fully and clearly defined by the user. Also see Av and Aw. The symbol Ad is only to be used when it is necessary to discriminate between Ad and Av. NOTE.— Aw is for general use for *area* of *web* reinforcement.
In general	Ae	= *area equivalent* to some given area, or *area* of an *equivalent* section, or *equivalent area*.
In Reinf. Con.	Ae	= *equivalent area* = area of concrete + m times area of reinforcement.
In general	Ag	= *gross area* which must be fully and clearly defined by the user. Also see An.
In general	Ai	= *inner area*, which must be fully and clearly defined by the user.
In R.C. pillars	Ai	= *inner area* = cross sectional *area* measured *inside* the binding. Ai is only to be used by those who regard the area outside the binding as part of the effective area. Also see A and Ap.
In general	An	= *net area* which must be fully and clearly defined by the user. Also see Ag.
In Reinf. Con.	An	= *net area* of the concrete in a section, *i.e.*, area exclusive of the area of reinforcement.
In R.C. pillars	Ap	= cross sectional *area* of a *pillar* measured from outside to outside. Ap is only to be used by those who regard the area inside the binding as the effective area. Also see A and Ai.

In general	As	=	*superficial area.* In round bars $As = \pi \cdot d \cdot l$. In the case of piles, square in section and in square sections generally, $As = 4d.l$ where d = diameter.
In R.C. struts	As	=	cross sectional *area* of *steel* below or to the *left* of the centroidal axis.
In R.C. struts	As'	=	cross sectional *area* of *steel* above or to the *right* of the centroidal axis.
In R.C. beams	Av	=	cross sectional *area* or *vertical* web members in a specified length of a beam, as defined by the user. Also see Ad and Aw. NOTE.—Aw is for general use for *area* of *web* reinforcement.
In R.C. pillars	Av		cross sectional *area* of *vertical* reinforcement in a pillar. (Also used for area of longitudinal reinforcement in struts and other compression members placed horizontally or at any angle.)
In R.C. beams	Aw	=	cross sectional *area* of *web* reinforcement in a given length of beam, as defined by the user. Also see Ad and Av, which can be used if discrimination is necessary.
In R.C. beams	a	=	*arm* of the resistance moment or lever *arm*, measured between centre of tension reinforcement and centre of compressive force in concrete.
In R.C. beams	a_1	=	*arm* ratio = $a/d/$.
In R.C. beams	ac	=	*arm* of the *compression* reinforcement measured from the centre of tension reinforcement to centre of compression reinforcement. (Also compare with at.)
In R.C. beams	ac_1	=	ratio of ac to d.
In R.C. struts	ae	=	*arm* of the *extreme* fibres below or to the *left* of the centroidal axis measured from centroid of the cross section. NOTE.—a is for general use in beams but the symbols ae, ae', as, as', etc., are special symbols for limited use in connection with reinforced concrete struts and other members which act partly as pillars and partly as beams.
In R.C. struts	ae'	=	*arm* of *extreme* fibres *above* or to the *right* of the centroidal axis.
In R.C. struts	as	=	*arm* of *steel below* or to the *left* of the centroidal axis.
In R.C. struts	as'	=	*arm* of *steel above* or to the *right* of the centroidal axis.
In R.C. beams	at	=	*arm* of *tension* reinforcement *measured from the neutral axis.* Also see ac in Reinf. Con. beam.
In general	α	=	*script alpha*—an *angle* as specified by the user.
In beams	B	=	*bending* moment of the external loads and reactions.
In pillars	B	=	*bending* moment due to eccentricity of load.
In general	Ba, Bb, Bc, etc.	=	a series of *bending* moments at consecutive cross sections when indicated on some diagram.

In beams	Bc	= *bending* moment at the *centre* of the span. (Bc is only used when B is not sufficiently explicit.)
In general	B	= *bending* moment of the forces to the left of a section. (If and when necessary.)
In general	B'	= *bending* moment of the forces to the *right* of a section. (If and when necessary.) NOTE.—$B = B'$.
In beams	Bs	= *bending* moment at the *supports* of a fixed beam. (Bs is only used when B is not sufficiently explicit.) NOTE.—Be for *bending* moment at the *end* of a fixed beam is not used because Be is more likely to be confused with Bc than Bs would be.
In beams	Bs	= *bending* moment over the *supports* of a continuous beam. (See Ba, Bb, etc.)
In beams	Bx	= *bending* moment at x. (Bx is only used when B is not sufficiently explicit.)
In general	b	= *breadth*.
In general	bo	= *breadth over all*. (Only used when b and B are already in use in the same equation.)
In beams	b	= *breadth* of a rectangular beam or *breadth* of a tee beam, measured across the compressed edge.
In general	bn	= *net breadth* as defined by the user.
In R.C. beams	br	= *breadth* of the *rib* of a tee beam.
In general	bs	= *buckling stress*.
In general	C	= a *constant* or *coefficient* as defined by the user.
In general	Ca, Cb, Cc	= a series of *constants* arranged in order, and described in the context.
In general	C_1, C_2, C_3	= a series of *constants*. NOTE.—that Ca, Cb, Cc, etc., should be used, and will be more easily typed than C_1, C_2, C_3.
In Reinf. Con.	C	= total *compression* on the concrete (when specially stated by the user).
In general	CG	= the *centre* of *gravity* (a point not a dimension).
In Reinf. Con.	Cs	= total *compression* in the *steel*.
In R.C. beams	c	= intensity of *compressive* stress in the concrete at the compressed edge of the beam.
In R.C. pillars	c	= intensity of direct *compressive* stress in the concrete.
In R.C. struts	c	= intensity of direct *compressive* stress in the concrete.
In Reinf. Con.	c_t	= c/t = the ratio of *compressive* stress to tensile stress.
In geometry	c	= *centroidal* distance, *i.e.*, the distance of the centroid of an area from some given axis
In general	ca, cb, cc, etc.	= a series of *centroidal* distances which have no specific names, but are indicated on some diagram.
In general	cc	= *circumference*. (This is in place of the former symbol for periphery.)

In general	*cf*	= cu. ft. = *cubic feet.*
In general	*ci*	= *cu. in.* = *cubic inches.*
In Reinf. Con.	*ci*	= intensity of *compressive* stress in the concrete, at the *intersection* of the slab and the rib in tee beams, when so defined by the user. (Used in place of *cu* which is now appropriated for *ultimate compressive* stress.)
In general	*cm*	= intensity of *mean compressive* stress.
In Reinf. Con.	*cs*	= intensity of the *compressive* stress in the *steel.*
In Reinf. Con.	*cu*	= intensity of *ultimate crushing* resistance of plain concrete or intensity of *ultimate compressive* stress in concrete.
In general	D	= *depth* greater than d, as defined by the user. (Only used when necessary to discriminate from d.)
In pillars	D	= *diameter* greater than d as defined by the user.
In pillars	D_1	= D/d = ratio of D to d.
In R.C. struts	D	= *diameter* greater than d, as defined by the user.
In general	d	= *depth.*
In R.C. beams	d	= effective *depth* of a beam, measured from the compressed edge to the centre of the tension reinforcement.
In circular sections	d	= *diameter.*
In pillars	d	= *diameter* generally; least *diameter* of a pillar when so defined by the user.
In R.C. beams	*db*	= *diameter* of a *bar.*
In R.C. pillars	*db*	= *diameter* of one bar of the *binding* or lateral reinforcement.
In general	*dc*	= *distance* between any two *centres.*
In R.C. pillars	*dc*	= *distance* between the *centres* of the vertical bars measured normal to the neutral axis.
In general	*di*	= *diameter* of an *inner* area.
In general	*dn*	= *deflection.*
In R.C. pillars	*dv*	= *diameter* of the bars of the *vertical* reinforcement.
In general	Δx	= *differential* of x.
In general	Δy	= *differential* of y.
In general	$\frac{dy}{dx}$	= limit of the ratio $\Delta x/\Delta y$.
In general	E	= *elastic* modulus of any material: Young's *elastic* modulus. NOTE, E will be in units of (Force/Area.) Usually in tons/sq. in. or pounds/sq. in.
In general	*Ec*	= *elastic* modulus of *concrete.*
In general	*Es*	= *elastic* modulus of *steel.*
In general	e	= *elongation.*
In pillars	e	= *eccentricity* of the load measured from the centre of the pillar to the point of application of the load.
In pillars	e_1	= *eccentricity* ratio = e/d = ratio of e to the diameter of the pillar measured in the direction of e.

In general	*ec*	= *elastic* limit of *concrete*.
In pillars	*cf*	= *end fixity* factor.
In general	*es*	= *elastic* limit of *steel*.
In general	*F*	= *force*: total force, etc.
In mechanics	*F*	= *friction*: total *friction*, where so defined by the user.
In beams	*F*	= total *flexural force* when it is not desired to discriminate between *C* and *T*. EXAMPLE: $B = R = F{\cdot}a$ and $F = C = T$. F = total *flange* load = total load on the *flanges* of steel girders.
In pillars	*F*	= a *form* factor = a factor for *form* of cross section in Gordon's formula. Also see *K* (in pillars).
In general	*Fa*, *Fb*, *Fc*, etc.	= a series of *forces* which have no specific names but are indicated on some diagram.
In R.C. pillars	*f*	= a *form* factor which will vary according to whether the binding is curvilinear, or rectilinear, etc.
In beams	*f*	= intensity of *flexural* stress = extreme *fibre* stress, *i.e.*, stress at the extreme *fibre* of any member under transverse load.
In beams	f_1	= *f* (subscript *one*) = *flexural* stress at *one* inch from the neutral axis.
In R.C. struts	*fc*	= intensity of *flexural* stress at the extreme edge of the *concrete below* or to the left of the centroidal axis.
In R.C. struts	*fc'*	= intensity of *flexural* stress at the extreme edge of the *concrete above* or to the right of the centroidal axis.
In general	*fs*	= *factor* of *safety*. Compare with *sf* = square feet.
In R.C. struts	*fs*	= intensity of the *flexural* stress in the *steel* below or to the *left* of the centroidal axis.
In R.C. struts	*fs'*	= intensity of the *flexural* stress in the *steel above* or to the *right* of the centroidal axis.
In beams	*fu*	= intensity of the *ultimate flexural* stress = *ultimate* "*fibre*" stress = modulus of rupture.
In general	φ	= *phi* = *friction* modulus. NOTE.—This letter can be typed by striking the solidus over a capital O.
In general	φk	= *friction* modulus for *kinetic* friction (when necessary to discriminate from φs.
In general	φs	= *friction* modulus for *static* friction (when necessary to discriminate from φk.
In general	*G*	= *geometrical* moment, *i.e.*, the first moment of an area about some specified axis.
In R.C. beams	*G*	= *geometrical* moment or first moment of the concrete compression area about the neutral axis.
In general	*Ga*, *Gb*, *Gc*, etc.	= a series of *geometrical* moments which have no specific names, but are indicated on some diagram.
In R.C. beams	*Ge*	= *geometrical* moment of an *equivalent* area as defined by the user.

In general	g	= *gravity* coefficient = 32·2 ft./sec.² = *gravitational* acceleration.
In dynamics	g	= *gravity* coefficient = 32·2 ft./sec.² = *gravitational* acceleration. NOTE.—In problems dealing with revolving flywheels and other revolving masses, g would be used for the *gravity* coefficient and gc would be used for the *gyration* radius about a *central* axis or gx for the *gyration* radius about the axis XX, R and r would be used for a greater and a lesser *radius* respectively.
In statics	g	= *generic* radius = *gyration* radius.
In pillars	g	= *gyration* radius where so defined by the user. In all other cases use gb, gc, ge, gr, gx, gy, gz, etc., to avoid confusion with g = *gravity* coefficient.
In R.C. beams	g	= intensity of *grip* stress between concrete and steel where so defined by the user. In all other cases use gs to avoid confusion with g = *gravity* coefficient.
In general	gb	= *gyration* radius about a *basal axis*, *i.e.*, an axis passing through the base of the area.
In general	gc	= *gyration* radius about a *central* axis.
In general	ge	= *gyration* radius about an *external* axis, *i.e.* an axis external to the area.
In general	gg	= *greatest gyration* radius. Only used when discrimination is necessary. Also see gz.
In general	gl	= *least gyration* radius. Only used when discrimination is necessary. Also see gw. NOTE.—g will generally suffice.
In general	gw	= *gyration* radius about the axis WW, where WW, XX, YY, ZZ, denote various axes passing through the centroid of the section at various angles. In sections which are not symmetrical about the rectangular axes XX and YY, the gyration radius about the axis WW should denote the *least* value while gz should denote the *greatest* value.
In general	gx	= *gyration* radius about the axis XX; *i.e.*, the horizontal axis.
In general	gy	= *gyration* radius about the axis YY, *i.e.*, the vertical axis.
In general	gz	= *gyration* radius about the axis ZZ. See note against gw.
In general	γ	= *gamma* = specific *gravity*.
In general	H	= *height* greater than h = total *height* when h is the height to centre of pressure.
In general	h	= *height* generally.
Occasionally	h	= *height* to centre of pressure when so defined by the user.
In piles	h	= *height* of fall of ram. (h will be in the same units as s in piles.)
In general	I	= *inert* moment = *inertia* moment of an area in (sq. inches × ins.²) or inches⁴.
In beams	Ib	= *inert* moment or *inertia* moment of a *beam*.

In statics	Ib	= *inert* moment or *inertia* moment about a *basal* axis when so defined by the user. EXAMPLE.—The inertia moment of a rectangle about an axis passing through its base = $Ib = b \cdot h^3/3$ where b = *breadth* and h = *height*.
In Reinf. Con.	Ic	= *inert* moment or *inertia* moment of *concrete* only.
In statics	Ic	= *inert* moment or *inertia* moment about a *central* axis when necessary to discriminate and when so specified by the user.
In R.C. pillars	Ie	= *inert* moment or *inertia* moment of an *equivalent* area.
In statics	Ie	= *inert* moment or *inertia* moment about an *external* axis, when necessary to discriminate and when so specified by the user.
In pillars	Ip	= *inert* moment or *inertia* moment of a pillars
In Reinf. Con.	Is	= *inert* moment or *inertia* moment of *steel* only.
In general	Ix	= *inert* moment or *inertia* moment on axi. XX, *i.e.*, the horizontal axis.
In general	Iy	= *inert* moment or *inertia* moment on axis YY, *i.e.*, the vertical axis.
In general	i	= *inset* = distance from an outer edge some point specified by the user.
In R.C. beams	i	= *inset* of reinforcement from the bottom or top of a member as specified by the user.
In R.C. struts	i	= *inset* of reinforcement from the bottom or the left of a member.
In R.C. struts	i'	= *inset* of reinforcement from the *top* or from the *right* of a member. NOTE.—If i is required to denote *increased* stress then ir and ir' would be used to denote the *insets* of *reinforcement*.
In general	i_r	= *inset* ratio = i/d, or i/D where specified by the user.
In R.C. pillars	i	= *increased* stress permissible by reason of the provision of specified percentages of binding.
In pillars	K	= coefficient for *kind* of material, in Gordon's & Rankine's formulæ. Also see F.
In general	κ	= *kappa* = *compressive* strain. Also see λ and τ.
In general	L	= *length* greater than l.
Occasionally	L	= *length* in feet (when specially stated by the user).
In piles	L	= total *length* = $l + la$.
In general	l	= *length*.
In beams	l	= effective *length*, *i.e.*, length between centres of bearings.
In piles	l	= *length* of pile below the ground in the same units as a and h.
In pillars	l	= *length* between centres of lateral supports.
In pillars	l_r	= l/d = ratio of *length* to least *diameter* of a pillar, *i.e.*, the slenderness of a pillar in terms of least diameter.

In general	*la, lb, lc,* etc.	=	a series of *lengths* which have no special names but are indicated on some diagram.
In piles	*la*	=	*length* of pile *above* the ground.
In beams	*lc*	=	*length* of *clear* span.
In pillars	*l/g*	=	ratio of *length* to *gyration* radius = slenderness of a pillar in terms of the *gyration* radius.
In pillars	*lv*	=	*virtual length.*
In general	λ	=	*lambda* = *longitudinal* strain which may be either tensile or compressive. Also see κ and τ. NOTE.—(Lambda) is occasionally used to represent *length* increment or decrement due to tensile or compressive stress, but this quantity would be more properly represented by Δl, since it is the *differential* of the *length*.
In dynamics	*M*	=	*momentum* = $(W/g)v$ = *m.v.*
In statics	*M*	=	*modulus* generally, including *modulus* of an area, *modulus* of a section, *modulus* of a beam or pillar (in square inches × inches), *i.e.*, in inches.³. NOTE.—$M = I/n$.
In Reinf. Con.	*M*	=	*modulus* of a section in terms of the arm of the extreme fibre below or to the left of the centroidal axis = Ie/ae where ae = arm of the *extreme* fibre below or to the left of the centroidal axis.
In Reinf. Con.	*M′*	=	*modulus* of a section in terms of the arm of the *extreme* fibres *above* or to the right of the centroidal axis. NOTE.—$M' = Ie/ae'$ where ae = *arm* of the *extreme* fibre above or to the right of the centroidal axis.
In Reinf. Con.	*Mc*	=	*modulus* of the area in *compression* = *modulus* of a beam in terms of the *compressed* area.
In Reinf. Con.	*Ms*	=	*modulus* of a section in terms of the arm of the *steel* below or to the left of the centroidal axis. NOTE.—$Ms = Ie/as$.
In Reinf. Con.	*Ms′*	=	*modulus* of a section in terms of the arm of the *steel above* or to the *right* of the centroidal axis. NOTE.—$Ms' = Ie/as'$.
In Reinf. Con.	*Mt*	=	*modulus* of the area in *tension* = *modulus* of the area of *tensile* reinforcement = $A \cdot a$.
In general	*Mx*	=	*modulus* of a section about the axis *XX*.
In general	*My*	=	*modulus* of a section about the axis *YY*.
In Reinf. Con.	*m*	=	*modular* ratio = Es/Ec = *modular* ratio for steel and concrete.
In dynamics	*m*	=	*mass* = W/g = *inertia* (dynamical).
In general	*N*	=	a *numerical* coefficient, or a *number*.
In general	*Na, Nb, Nc*	=	a series of *numerical* coefficients *Na, Nb, Nc*, are more easily typed than N_1, N_2,

In general	Nb	= *number of bars.*
In general	Nd	= *number of divisions.*
In general	n	= the *norm* = the *length of a normal* ≡ the length of a line measured at right angles to some other line.
In beams	n	= length of the *normal* to the *neutral* layer = the distance measured from the *neutral* layer to the extreme fibre.
In beams	na	= length of the *normal above* the neutral layer (when necessary).
In beams	nb	= length of the *normal below* the neutral layer (when necessary).
In beams	nc	= length of the *normal* on the *compression* side of the beam (when necessary).
In beams	nt	= length of the *normal* on the *tension* side of a beam (when necessary).
In general	n_1	= n/d = the *norm* ratio = the *neutral* axis ratio. Also $n_1 = n/D$ if and when specified by the user.
In beams	n'	= the value of n when measured *above* the neutral axis, and when n is measured *below* the neutral axis, but na and nb or nc and nt are preferable to accented letters.
In beams	n'_1	= n'/d (when necessary, but na_1 and nb_1 are preferable.
In general	P	= total *pressure.*
In pillars	P	= total *pressure.*
Occasionally	P	= *push* or *pull* when so specified by the user.
In silos	Pb	= total *pressure* on the *bottom* of a silo.
In Reinf. Con.	Pc	= total *pressure* on *concrete.*
In general	Ph	= total *horizontal pressure.*
In general	Pn	= total *normal pressure.*
In pillars	Ps	= total *pressure* on a *short* pillar when specified by the user, and when P = total *pressure* on a long pillar.
In Reinf. Con.	Ps	= total *pressure* on *steel.*
In silos	Ps	= total *pressure* on the *side* of a silo.
In general	Pt	= total *tangential pressure.*
In general	Pv	= total *vertical pressure.*
Generally	p	= intensity of *pressure.*
Occasionally	p	= *projection* when so specified by the user.
In R.C. pillars	pb	= the *pitch* of the *binding* (*i.e.*, the axial spacing of the laterals).
In R.C. pillars	p_1	= pb/d = *pitch* ratio = ratio of *pitch* of the *binding* to diameter of hooped core.
In silos	pb	= intensity of *pressure* on the *bottom* of a silo.
In Reinf. Con.	pc	= *percentage* of reinforcement. Example: $pc/100$ = ratio and pc = $100 \times$ ratio.
In general	ph	= intensity of *pressure* acting *horizontally.*
In general	pn	= intensity of *pressure* acting *normal* to some specified surface.
In silos	ps	= intensity of *pressure* on the *side* of a silo.
In general	pt	= intensity of *pressure* (*tangential*).
In general	pv	= intensity of *pressure* acting *vertically.*

In Reinf. Con.	*pw*	= *pitch* of *web* members measured horizontally (*i.e.*, distance apart centre to centre). NOTE.—*pd* and *pv* can be used for *pitch* of *diagonal* web members and *pitch* of *vertical* web members if discrimination is necessary.
In general	*II*	= greater *pi* = *Poisson's* ratio.
Universally	π	= lesser *pi* = *peripheral* ratio of a circle = ratio of the circumference of a circle to its diameter = 3·1416.
In general	*Q*	= any *qualifier* or algebraical *quantity* used as a multiplier or coefficient as specified by the user ; also a *quotient*.
In beams	*Q*	= a *qualifier*. Example : $R = Q{\cdot}b{\cdot}d^2$. $R/b{\cdot}d^2 = Q$.
In general	*q*	= any *qualifier* or algebraical *quantity* used as a multiplier, or coefficient, as specified by the user ; also a *quotient*.
In general	*R.C.*	= *reinforced concrete*.
In general	*R*	= *resistance* moment, *R* will be in units of (Force × length). Example : pound-inches.
In Reinf. Con.	*R*	= *resistance* moment at the critical percentage of reinforcement (when specially stated by the user).
In general	*Rc*	= *compressive resistance* moment = resistance moment in terms of the compressive stress. (Only used when necessary to distinguish from *Rt*.)
In general	*Rt*	= *tensile resistance* moment or *resistance* moment in terms of the *tensile* stress. (Only used when necessary to distinguish from *Rc*.)
In piles	*R*	= working statical *resistance* of the ground to further penetration by the pile. (*R* will be in the same units as *W* and *Wp*.)
In mechanics	*R*	= *resultant* of a series of forces when so specified by the user.
In mechanics	*Ra*, *Rb*, *Rc*	= a series of *reactions*, counted from the left towards the right, only used in connection with some explanatory diagram.
In general	*r*	= *radius*.
In general	*ra*, *rb*, *rc*	= a series of *radii* indicated on some diagram.
In Reinf. Con.	*r*	= *ratio* (when specially stated by the user).
In R.C. beams	*r*	= *ratio* of area of tensile reinforcement to the area *b·d*, *i.e.*, $r = A/b{\cdot}d$.
In R.C. struts	*r*	= *ratio* of area of reinforcement *below* or to the *left* of the centroidal axis to area of cross section of strut.
In R.C. struts	*r′*	= *ratio* of reinforcement *above* or to the *right* of the centroidal axis when specified by the user.
In R.C. pillars	*r*	= *ratio* of vertical reinforcement to effective area . $r = Av/A$.
In piles	*r*	= *ratio* as may be specified by the user. Example : $r = la/d$.

In R.C. beams	*r*	= *ratio* of area of *compressive* reinforcement to area of beam, *i.e.*, $r = Ac/b \cdot d$.
In R.C. struts	*rc*	= *resultant* stress at the extreme edge of *concrete below* or to the *left* of the centroidal axis.
In R.C. struts	*rc'*	= *resultant* stress at the extreme edge of the *concrete above* or to the *right* of the centroidal axis.
In R.C. struts	*rs*	= *resultant* stress in steel
In R.C. struts	*rs*	= *resultant* stress in *steel* below or to the left of the centroidal axis.
In R.C. struts	*rs'*	= *resultant* stress in *steel* above or to the right of the centroidal axis.
In steel work	*rr*	= *radius* at the *root* of a section.
In steel work	*rt*	= *radius* at the *toe* of a section.
In beams	*S*	= the total *shear* at a given vertical section
In arches	*Sa*	= total *shear* at the *abutments*.
In arches	*Sc*	= total *shear* at the *crown*.
In R.C. beams	*s*	= intensity of the *shearing* stress on concrete.
In piles	*s*	= average *set* of a pile under last few blows. (*s* will be in the same units as *h*.)
In pillars	*s*	= *spacing* factor or constant which will vary with the pitch of the binding.
In general	*sf*	= *sq. ft.* = square feet.
In general	*si*	= *sq. in.* = square inches.
In general	*sn*	= *strain*, when *ss* = stress.
In general	*ss*	= *stress* intensity, when *sn* = strain.
Occasionally	*ss*	= *shearing* stress in the *steel* when so defined by the user.
Universally	Σ	= *sigma* = the *sum* of.
In R.C. struts	Σs	= *sum* of the *stresses*.
In general	σ	= *sigma* = *shearing* strain.
In general	*T*	= total *tension* in the steel.
In general	*Ta*, *Tb*	= total *tensile* forces at consecutive sections indicated on some diagram.
In arches	*Tc*	= *thrust* at the *crown* of an arch.
In R.C. beams	*Td*	= *tensile* force (*diagonal*) or total *diagonal* tension.
In Reinf. Con.	*t*	= intensity of *tensile* stress on the steel.
In R.C. struts	*t*	= *tensile stress* intensity due to flexure.
In R.C. beams	*tc*	= *tensile stress* intensity in *concrete*.
In beams	*td*	= intensity of *diagonal tensile* stress.
In general	*th*	= *thickness*.
In steelwork	*tf*	= *thickness* of *flange*. Also see θ.
In general	*tw*	= *thickness* of *web*.
In general	τ	= *tau* = *tensile* strain. Also see [illegible] and [illegible].
In R.C. tee beams	θ	= *theta* = *thickness* of the slab which forms the flange of a reinforced concrete tee beam. Also see *tf*.
In R.C. tee beams	θ,	= *thickness* ratio = θ/d.
In general	*U*	= *unital* constant; a constant employed to rationalize a set of units.
Occasionally	*U*	= total *ultimate* load when specified by the user. (Compare *W* = working load.
In general	*V*	= *volume*.

In R.C. pillars	V	= *volume* of a pillar or portion of a pillar, as defined by the user.
In R.C. pillars	Vb	= *volume* of *binding* or volume of portion of the binding as defined by the user.
In R.C. pillars	V_r	= Vb/v = *volumetric* ratio = the ratio of *volumes*, *i.e.*, the ratio of the volume of binding to the volume symbolized by V.
In arches, etc.	v	= *versin.*
In general	v	= *velocity.*
In general	va	= *angular velocity.*
In general	vi	= *initial velocity.*
In general	vu	= *ultimate* or final *velocity.*
In general	W	= total *weight.*
In beams	W	= total *working* load or *weight* on a beam (including its own weight). NOTE.—$W = Wd + Ws$.
In beams	Wd	= total *dead weight.*
In beams	Ws	= total *superimposed weight.*
In piles	W	= *weight* of the ram.
In piles	Wp	= *weight* of the pile.
In general	w	= *weight* or *working* load per unit of length or area as stated by the user.
In beams	wd	= intensity of *dead weight* in units of (Force/Area).
In beams	ws	= intensity of *superimposed weight* in units of (Force/Area), usually in pounds per square foot.
In steelwork	Y	= *yield* load or total load at *yield* point (of metals) in units of Force.
In steelwork	y	= intensity of *yield* stress or stress at the *yield* point of a metal in units of Force/Area.

PART II

INTERNATIONAL IDEOGRAPHS, ETC.

(Note, as a matter of convenience to authors, these ideographs have been described in terms used by printers in preference to the terms used by mathematicians, especially in those cases where the signs have more than one name.)

(1) ANGLES

Script alpha = any angle in any position.
The angle mark. Example: $\alpha = \angle \sin y$, that is: alpha is the angle whose sine is y. Conversely, y sin α.
Lower case theta = the angle of inclination to the *horizontal* plane. Note the *horizontal* direction of the mnemonic straight line in this symbol.
Lower case psi = the angle of inclination to the *vertical* plane. Note the *vertical* direction of the mnemonic straight line in this symbol. Note also that angles are frequently repeated in various positions on a diagram, and in some positions φ (phi) might be confused with θ (theta). Therefore φ is discarded as a symbol of the angle of inclination to the vertical plane and is replaced by ψ (psi). Note $\theta + \psi = 90°$.
The degree mark.
The minute mark.
The seconds mark.

(2) BRACKETS AND BRACES

The round brackets: the first sign of aggregation.
The square brackets: the second sign of aggregation.
The braces: the third sign of aggregation.

Note that the braces should enfold the square brackets, and the square brackets should enfold the round brackets.

Thus: { [( )] }.

Brackets or braces should be used in place of the vinculum (‾) as the horizontal bar of the vinculum increases the cost of typesetting out of all proportion to its value as a sign of aggregation.

The standard notation does not include the vinculum, and the increasing use of typesetting machinery will, in any case, lead to its general abandonment.

(3) DASHES

The minute mark.
The seconds mark.
The thirds mark.

Note that the above dashes when used in conjunction with *numerals* always denote quantities of three different orders of magnitude, but the meaning in any particular case will depend upon the context.

($'$) The bold minute mark when used in conjunction with a *letter of the alphabet* always denotes some quantity *to the right of*, or *above* some axis of reference.

Example. If d denotes a distance to the left of a vertical axis, then d' denotes a distance to the right of that axis. If d denotes a distance *below* some horizontal axis, then d' denotes some distance *above* that axis.

($\backslash$) The contra-italic dash used in conjunction with a letter, indicates one of a series of ratios having a common consequent.

Example. If $c \equiv$ the common *consequent*, then the ratios a/c, b/c, c/c, d/c, e/c, etc., would be represented by $a\backslash$, $b\backslash$, $c\backslash$, $d\backslash$, $e\backslash$, etc.

! Factorial mark. Example : $4\,! = 1 \times 2 \times 3 \times 4$.

This symbol was introduced by Kramp in 1808, and it is much easier for the printer than that L-shaped factorial mark which necessitates a horizontal line being put under the numeral. To distinguish Kramp's factorial mark from others it can also be described as the "bold italic exclamation mark."

(4) POINTS

(˙) The high point is the decimal separatrix.

Example : 5˙7 = five plus seven-tenths.

(·) The middle point is the sign of multiplication for use between literal symbols.

Example : $m{\cdot}n = m$ multiplied by n.

(.) The low point is the sign of abbreviation used in the current text, but not in equations.

Example from current text B.H.P. = Brake Horse Power.

Example from an equation, BHP = Brake Horse Power.

(:) The ratio mark.

(...) The continuation mark denoting "and so on."

(∵) The "because" mark.

(∴) The "therefore" mark.

(: :) The twin colons denoting the equality of two *abstract numbers*.

(5) SIGNS OF EQUALITY AND VARIATION, ETC.

= The equality mark, denoting "is equal to." This is the general sign of equality for use in connection with *numbers* or *magnitudes*. Compare with the special sign (: :) which is used in connection with abstract *numbers* if and when discrimination is necessary for some special purpose.

≑ The approximate equality mark, denoting "is approximately equal to." (A point above and below equality.)

≡ The identity mark. This sign denotes something more than numerical equality, viz., *identity*. It is frequently used between two different symbols when they both denote the self same quantity.

For example—

$$\boldsymbol{Ab} \equiv \textbf{\textit{Area}} \text{ of the } \textbf{\textit{beam}}.$$

Compare this with

$$\textbf{Area of the beam} = \textbf{12 sq. in.}$$

Or

$$\boldsymbol{Ab} = \textbf{12 sq. in.}$$

$\neq$ The inequality mark, denoting " is not equal to."

$<$ The less mark, denoting " is less than."

$\nless$ The not-less mark, denoting " is not less than."

$\leqq$ Marks denoting " is less than or equal to." This full form is preferable to the special sign ($\leq$) which might be discarded without great loss.

$>$ The greater mark, denoting " is greater than."

$\ngtr$ The not greater mark, denoting " is not greater than.

$\geqq$ Marks denoting " is equal to or greater than." This full form is preferable to the special sign $\geq$ which might be discarded without great loss.

« The congruence mark, denoting " is congruent with." Large bold French " guillemets " can be used for this purpose. Note that this symbol vividly conveys the idea of congruency.

$\sim$ The difference mark, denoting the difference between two quantities irrespective of any question of positive or negative sign.

$\propto$ The variation mark, denoting " varies with." Thus, if y is a function of x, then $y \propto x$, *i.e.*, y varies with x.

$\rightarrow$ The arrow mark, denoting " approaches." Thus $\Delta x \rightarrow 0$, *i.e.*, the differential of x approaches zero.

(6) SIGNS OF OPERATION AND OPPOSITION, ETC.

$+$ The plus mark denoting—

(*a*) addition.
(*b*) positive quantities.
(*c*) compressive forces

as specified by the user.

$-$ The minus mark denoting

(*a*) subtraction.
(*b*) negative quantities.
(*c*) tensile forces

as specified by the user.

$\pm$ The plus-minus mark denoting $+$ or $-$.

$\times$ The multiply mark, used as the sign of multiplication between numerals.

($\cdot$) The middle point used as the sign of multiplication between literal symbols.

Example : $m \cdot n \equiv m$ multiplied by n.

Also see the symbols under the heading of " points."

$\div$ The divide mark, denoting " divided by."

(:) The ratio mark, which separates the antecedent from the consequent.

Thus, if $a =$ the *antecedent*, and $c =$ the *consequent*, then $a : c$ represents the ratio of a to c.

The bold solidus, denoting, either division or ratio according to the requirements of the context.

(NOTE.—Authors must in every case specify the *bold* solidus, or the printer will generally insert a solidus much too faint and weak. Note also that the horizontal bar between the numerator and the denominator of a fraction is much more costly to print than the solidus. Therefore the horizontal bar should not be used if a bold solidus will give equal clarity and vividness.

The root mark. Thus $\sqrt{2}$ = the square root of 2. Note the absence of the horizontal bar over the figure 2. The vinculum cannot be set up by machinery, and it also disturbs the even spacing of the context. Therefore it should not be used in conjunction with the root mark. Thus, write $\sqrt{(a \cdot b \cdot c)}$ and not $\sqrt{a \cdot b \cdot c}$. Also see note under the heading of Brackets and Braces.

(7) SIGNS OF QUANTITY

O The zero mark, the nullo.

$\dot{O}$ A quantity on the verge of nothing (a point above zero).

Q Any quantity. Example: $Q \rightarrow O$ = a quantity approaching zero, a diminishing quantity.

∞ The infinity mark.

REINFORCED CONCRETE

CHAPTER I

ORIGIN AND GENERAL CHARACTERISTICS

Early History.—Statements are still occasionally made to the effect that reinforced concrete is "a new and untried material." The fact is, however, that the principles of reinforced concrete were understood by the Ancient Romans, by whom the material was employed in the construction of roofing over monumental tombs and in important public buildings.

The reinforcing bars were of bronze and the concrete was made of lime mixed with volcanic scoriæ, and aggregate consisting of rather large fragments of broken stone. In some cases, iron bars were employed as reinforcement, or clamps, as, for instance, in the Parthenon, where the bars were found to be in good condition after having been embedded in mortar joints for a period of fully 2,000 years. In other cases, the Romans used timber as reinforcement, thus anticipating a patent taken out in quite recent times.

Although early Roman cement was far inferior to modern Portland cement, many examples of ancient construction are still extant, proving the remarkable durability of concrete as a structural material.

The Pont du Gard, in the South of France, built about 56 B.C., is a striking illustration of concrete bridge building, faced with stone after the custom of the Romans. A drawing of this structure is reproduced as Fig. 1. The dome of the Pantheon, erected A.D. 123, and shown in section by Fig. 2, is probably the finest example of ancient concrete work extant. Having a diameter of 142 ft., with an opening of 30 ft. diameter at the top, this structure has successfully

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resisted the destructive influences of the weather for nearly 2,000 years.

A concrete floor in the House of the Vestals, with a span of 20 ft., and a thickness of 14 in. and the Aqueduct of

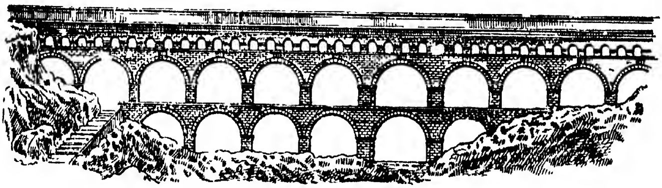

FIG 1

Vejus, are further examples of early concrete construction, others being found in the remains of ancient buildings in Greece, Mexico, and elsewhere.

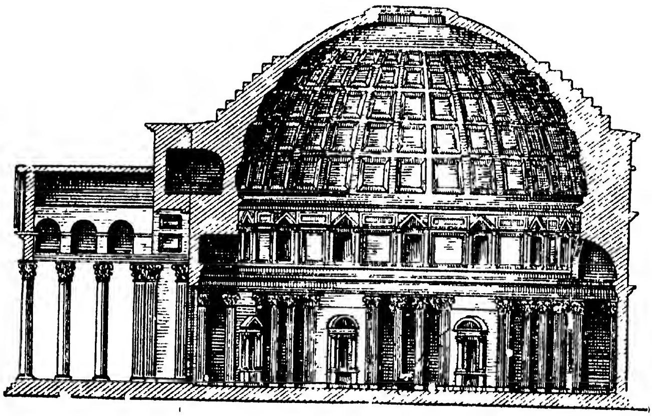

FIG. 2

During the reign of Julius Cæsar, concrete became quite common for foundations and the massive parts of masonry buildings, and some structures of that period are to be found consisting entirely of concrete faced with stone. In the time of Augustus, concrete construction became

practically universal for foundation work, the massive part of buildings, and the core of walls. The same material was also largely used for the construction of sewers, water mains, aqueducts, bridges, and highways.

It appears that the method of applying concrete was very much akin to that now generally adopted, for, as described by Palladio in 1570, the Romans used boards laid on edge to form moulds or shuttering, and filled the intervening space with cement and all sorts and sizes of stones mingled together.

Many Roman remains in Great Britain prove the durability of ancient concrete and mortar, a later example being furnished by the walls of the Benedictine Abbey at Reading, founded by Henry I in 1121. In this case, the walls appear to have been of stone with a concrete core, the latter still remaining, although the stonework has long since disappeared.

With the fall of the Roman Empire, the manufacture of cement was discontinued, and nothing seems to have been done in the way of reinforced concrete for several centuries.

One of the most interesting examples of reinforced concrete in mediæval times was reported from Paris in the year 1911. According to this account, the installation of a lift in the old Palace of the Louvre, rebuilt about A.D. 1527 by Francis I, led to the discovery that some of the ancient walls consisted of a kind of reinforced concrete, with a mere facing of dressed stone. Search in the archives has revealed the fact that the documents relating to the reconstruction carried out by Francis I make no mention of concrete, and clearly specify cut stone as the material for employment throughout the walls. Hence, we may reasonably assume that the economy of reinforced concrete was recognized by the contractors who rebuilt the Louvre nearly 400 years ago, although their employers were not invited to share in the saving effected.

Modern Development.—Coming to comparatively recent times, reinforced concrete flat roofs were suggested in 1830 by Loudon, and about 1840 two systems of reinforced plaster floor construction were employed, the methods of

reinforcement being very similar to some of those adopted to-day. In 1849, Lambot built the first reinforced concrete boat (Fig. 3), which was shown at the Paris Exposition of 1855 and is still in service. In 1855, Wilkinson took out the first British patent for a reinforced concrete floor, and in the same year Coignet patented his system of reinforcement. In 1861, Monier commenced the construction of boxes and vessels for horticultural use, and his system was developed later for application to arches, vaulting, floors, and other forms of construction.

The first reinforced concrete building in the United States

FIG. 3

was one in New York, erected in 1875 by Mr. W. E. Ward, the walls, floor beams, and roof being of concrete reinforced with metal. The first reinforced concrete bridge in the same country was built in 1892.

Among the numerous patentees who had appeared on the scene between 1855 and 1892, Hennebique, of Paris, deserves special mention as a great organizer, who established methods of calculation based upon extensive experimental researches, systematized forms of construction, and founded an organization of technical agencies and licensed contractors, which extends to almost every country in the civilized world, and

has been responsible for the execution of over 50,000 works at the present date.

Introduced into this country about 1896 by the late L. G. Mouchel, the Mouchel-Hennebique system of reinforced concrete is now represented by some 2,500 works in the United Kingdom, where numerous patented systems of reinforcement and special types of reinforcing bars or network have made their appearance during the past ten or fifteen years.

In the absence of complete data, it would be impossible to make any reliable estimate of the total number of reinforced concrete structures completed in this and other countries, but bearing in mind the activities of constructors in every part of the world, it is probable that the sum total runs into millions, and in any event must be large enough to negative any suggestion that reinforced concrete is an untried material.

On the question of antiquity, those who estimate the value of constructive methods by an archæological standard may find some assurance in the early history of reinforced concrete and in the fact that in modern times this combination was being employed on a practical scale at a period when cast iron and wrought iron were the only other substitutes for timber in the construction of beams, columns, and framed structures, and mild steel was quite unknown.

General Characteristics.—The designation *reinforced concrete* now generally adopted by British and American engineers is not quite appropriate, for the reason that it merely conveys the idea of concrete reinforced by the addition of steel, and ignores the fact that the steel is also reinforced by the concrete. Moreover, it altogether fails to express the combination of two materials, possessing equally valuable properties, in such a way as to enable them to act jointly in opposing resistance to external forces.

The co-equality of the partnership was denoted by the designation *concrete-steel*, originally applied but now practically obsolete, and is clearly suggested by the term

ferro-concrete, which is still very largely used by engineers and the technical press.

The properties of reinforced concrete and the behaviour of its constituent materials are discussed in subsequent chapters, and it is only necessary at the present stage to indicate very briefly the general characteristics of the combination.

In the first place, concrete is a material which the experience of thousands of years has proved to be unrivalled in respect of permanent strength, and resistance to the process of natural decay affecting all other structural materials to their ultimate destruction.

In structures where the principal effect of the external forces is the development of compressive stress, concrete is at its best. Possessing high resistance to compression, and capable of effective resistance to tension within reasonable limits, concrete construction is greatly superior to masonry and brickwork, whose stability depends mainly upon the force of gravity, aided to a small extent by the mortar joints.

On the other hand, concrete is not an economical material for forms of construction where the principal effect of the external forces is the development of tensile stress, or the development of compressive and tensile stresses in approximately equal proportions. The reason is that, like cast iron, concrete is considerably weaker in tension than in compression.

There is no doubt that the early use of reinforced concrete by the Romans was due to an intelligent recognition of this fact, which, in like manner, accounts for the modern development of the same material.

Although a very new material in comparison with concrete, structural steel is entirely reliable in respect of strength, its only weak point being an extreme susceptibility to corrosive influences. However, when embedded in concrete, the steel is effectively protected or reinforced against corrosion, and, thanks to the aid so furnished, is qualified for employment in permanent construction, for the purpose of

taking an assigned share in the duty of resisting the stresses developed by the application of external forces.

Reinforced concrete may be defined as a combination of concrete and steel wherein each material is applied so that its distinctive properties are utilized to the best advantage. The principles involved are most aptly illustrated by the case of a beam.

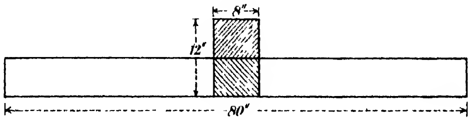

FIG. 4

If a rectangular beam were made of plain concrete, the tensile resistance of which is approximately one-tenth of its compressive resistance, the permissible working stress in tension would impose a corresponding limit on the compressive stress in the area above the neutral axis.

Thus, nine-tenths of the strength of the concrete acting in compression would necessarily be wasted. If no practical objections existed, this drawback might be obviated by adopting a tension flange ten times the width of the compression portion of the beam, in accordance with the principle followed in the design of cast iron beams. Fig. 4 represents the modified cross section of a plain concrete beam, originally measuring 8 in. broad by 12 in. deep, of the proportions that would be necessary for developing the compressive resistance of the material.

Such an arrangement would, obviously, be impracticable, and the same result could be obtained by inserting a small proportion of steel in the original tension area, thereby permitting the full compressive resistance of the concrete to be developed. Fig. 5 represents the cross section of a reinforced concrete beam of the same strength as the modified plain concrete beam shown in Fig. 4, and makes clear the great saving of concrete effected by the employment of steel in the tension area. In this diagram the three bars are placed

in a position where their centres are on the bottom line of the original cross section, 8 in. broad by 12 in. deep, as before, and a small amount of concrete is assumed to be added (as indicated by broken lines) for the purpose of covering the steel.

The economy resulting from the employment of the concrete for resistance to compression and the steel for

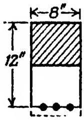

FIG. 5

resistance to tension may be gathered from the statement that for equal duty the cost of concrete in compression is only three-fifths the cost of steel, and the cost of steel in tension is only one-sixth the cost of concrete.

In the practical design of reinforced concrete beams, the tension bars are supplemented by steel in the form of web members for resistance to secondary stresses, and the employment of compression reinforcement is frequently desirable for the purpose of limiting the dimensions of members, or of providing for the reversals of stresses which take place in continuous beams under certain conditions of loading.

The employment of reinforced concrete in columns and other compression members is not attended with so much economy as that secured in the case of beams, because the steel is employed chiefly for resistance to compressive stress. In the case of short columns under axial loading, the two materials act jointly in withstanding direct compression, and in the case of long columns, and columns under non-axial loading, the two materials share the duty of resisting direct and bending stresses combined.

From this brief explanation, it will be seen that reinforced concrete differs essentially from steelwork merely encased in concrete for protection against fire or corrosion, and

that reinforced concrete must be considered as a combination possessing distinctive characteristics of its own.

Structural Advantages.—Strength and elasticity are two marked characteristics of reinforced concrete, which also possesses the valuable property of increasing in strength and durability with age, and combines the structural advantages of rigidity, impenetrability, impermeability, resistance to fire, and economy both as regards first cost and the elimination of the maintenance charges necessary in the case of all other structural materials.

Other advantages are to be found in the rapidity with which works can be executed, and in the fact that the constituent materials are readily obtainable in all districts, the steel requiring no preliminary workmanship, and the sand and aggregate for the concrete being frequently available in the vicinity of the works to be executed.

Distinctive Structural Features.—Reinforced concrete structures differ from those erected in accordance with ordinary methods in the respect that the entire fabric is of monolithic character, the concrete being in perfect connection throughout, and the steel affording a further assurance of continuity and mutual action between the various members or structural elements. Therefore, any well-designed and properly built reinforced concrete structure is capable of acting as a single unit in case of emergency, and if unexpected local stresses are suddenly developed by earth movements or any accidental cause, the integrity of the structure is maintained in consequence of the aid which is rendered by contiguous structural members to the part affected.

Another distinctive feature characterizing all reinforced concrete structures is that of lightness or slenderness, in comparison with the heaviness or massiveness of plain concrete, brick, and masonry structures. Thanks to the elastic strength of the combination, the engineer is enabled to depart from the custom of employing material in heavy masses and to adopt forms of design akin to those found in steel construction. Apart from the gain of interior space in buildings, reservoirs, and structures intended for storage

purposes, important economies are effected by this method of design. The saving of material in the superstructure naturally has the effect of reducing the load to be transmitted to the earth, thereby reducing the cost of foundations, and obviating the difficulties which frequently occur in work of the kind.

An attribute of modern reinforced concrete construction which should not be overlooked is that its fundamental basis is a method of scientific calculation complying more closely with theoretical principles than methods employed in other branches of constructive work. This view is appropriately expressed in a paper on "The Art of Construction," read to the French Society of Civil Engineers by Lieut.-Colonel Rabut, who says: "More happy than metallic construction, of which the fundamental types have been condemned by auscultation, reinforced concrete types have been modelled upon the results of experience. Moreover, contrary to an opinion widely accepted, the calculation of resistance for reinforced concrete structures is far more exact than that for metallic construction."

"Reinforced concrete has also led to another phase of scientific evolution wherein in place of basing preliminary calculations upon arbitrary assumptions, account is taken *a priori* of well-established values of the internal stresses. The general consequence of this innovation is the reduction of calculations of resistance to static calculations of simple and certain character."

CHAPTER II

CONSTITUTION AND PROPERTIES OF CONCRETE

Definition and Constitution.—Concrete in general may be defined as a variety of artificial stone, very much akin to the natural rock known as *conglomerate*, or *pudding-stone*, made by mixing either gravel or fragments of stone, or fragments of some other hard, inert substance, with cementitious material capable of binding the whole together to form a firmly cohering mass. The similarity of the two materials will be seen on comparison of Figs. 6 and 7, the former representing a piece of pudding-stone and the latter a fragment of concrete.

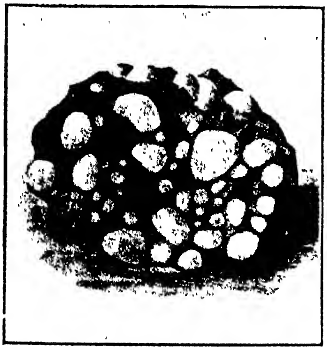

FIG. 6 -Conglomerate

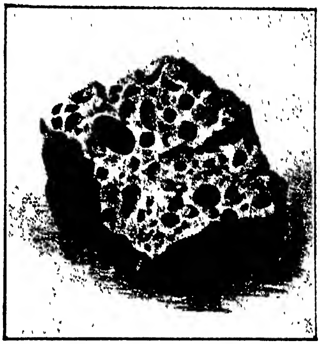

FIG. 7 -Concrete

In reinforced concrete practice, the use of gravel or broken stone is necessary in order to produce concrete of adequate strength and durability, and, for the same reason, Portland cement is employed as the cementitious material, instead of any of the varieties of lime and inferior cement which are still largely used in ordinary construction.

Where gravel forms the basis of concrete, this constituent is divided into (1) *sand*, comprising all the finer particles below a diameter of about $\frac{1}{8}$ in.; and (2) *aggregate*, consisting

of all the coarser particles, or pebbles, ranging from about $\frac{1}{8}$ in. to $\frac{3}{4}$ in. in diameter, or to any other limit that may be fixed. Where broken stone is used as aggregate, or coarse material, the finer particles necessary may consist either of sand or of stone screenings, or of a mixture of both materials.

The particles constituting the aggregate should be of various sizes, so as to reduce to a minimum the spaces, or *voids*, between them, and the particles of sand or stone screenings should be similarly graded in size with a like object.

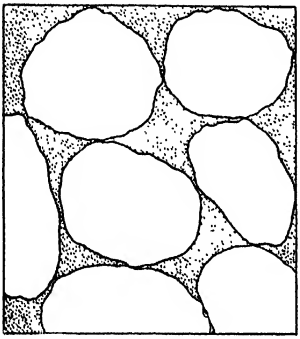

FIG. 8

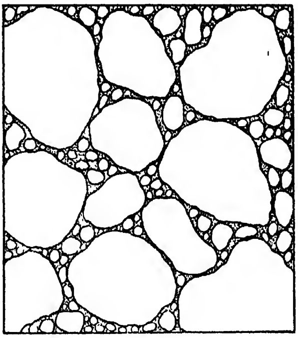

FIG. 9

If each class of material is properly graded, and if both classes are mixed together in correct proportions, the total amount of the voids in the mixture can be reduced to considerably less than half the average amount previously existing in the two constituents.

This point is illustrated by Figs. 8 and 9, one representing aggregate consisting of fragments of practically uniform size, and the other showing the manner in which the particles of well graded aggregates arrange themselves so as to reduce the proportion of the voids. These diagrams are equally applicable to sand and stone screenings, which only differ from aggregate in the respect that the particles are much smaller in size.

The voids remaining are filled by cement paste, formed by the addition of water to dry cement. The voids originally existing in the latter are eliminated in part by the reduction of volume following the admixture of water, the amount of cement paste being about 0·85 cubic foot per cubic foot of dry cement, and in part by a series of chemical reactions causing the cement paste to set in a hard and compact crystalline mass, the water absorbed in the formation of crystals becoming a permanent constituent of the concrete.

One important fact to be borne in mind is that there must always be an excess of cement paste to the extent of at least 10 per cent. over the voids in the sand, and an equal excess of mortar (cement paste and sand) over the voids in the aggregate, to allow for uneven mixing and losses in mixing, and to make sure that all the particles of sand and aggregate shall be thoroughly coated with cement.

The strength and other important properties of concrete depend very largely upon the care taken in the grading of the sand and aggregate, and upon the correctness of the proportions in which all the constituent materials are employed.

It may be added that the properties of concrete also depend upon the quality of the materials, thorough mixing, and proper treatment before, during, and after deposition in the work to be executed.

Limes and Cements.—While differing widely in respect of strength and physical properties generally, limes and cements are in reality members of one large family, being produced by the calcination of limestone or chalk, with or without the addition of foreign substances.

RICH or FAT LIME is produced by burning pure or nearly pure limestone or chalk, the resulting *quick-lime* slaking rapidly when mixed with water, swelling up and evolving great heat, and forming *slaked lime*.

POOR LIME is obtained by calcining limestone or chalk containing inert matter, the product slaking more slowly and less completely than rich lime.

HYDRAULIC LIME is made by calcining at moderate temperature limestone or chalk containing clay, either alone or in combination with alkalies and metallic oxides, in proportions sufficient to render the resulting material capable of setting under water. If the proportion of clay is relatively large, a variety of *natural cement* can be produced by continuing the process of calcination until the whole of the lime has entered into combination. The line of demarcation between hydraulic lime and natural cement is most conveniently drawn by classifying as *lime* those products which contain uncombined lime capable of slaking in the presence of water, and as *cement* those in which all the lime has entered into chemical combination, thus permitting the process of setting to take place without the slaking characteristic of lime.

Common lime can be rendered hydraulic by the addition of substances either before or after calcination. For instance, *Scott's* or *selenitic cement* consists of lime containing a small proportion of a sulphate, usually calcium sulphate, and the hydraulic lime or cement used by the Romans was produced by the admixture of *pozzuolana* with ordinary lime. Pozzuolana, trass, and similar natural products are varieties of argillaceous earth calcined by volcanic heat, and it is interesting to note that they are used to-day practically in the manner adopted by the Romans.

NATURAL CEMENT was first made in this country from septaria or nodules of clay and calcareous matter found at various points on the coasts of Kent and Essex. Originally known as *Parker's cement*, this product was afterwards named *Roman cement*, a superior quality of the same material being produced under the name of *Medina cement*, so called from the river in the Isle of Wight where septaria are also found.

Natural cement is produced by the calcination of rock containing varying proportions of limestone and clay. The chemical composition varies considerably even in the same quarry, and as the rock is burned as quarried, without preliminary adjustment of the proportionate amounts of lime

and clay, the inevitable result is a product of inferior and unreliable character.

Before the outbreak of the war in 1914, large quantities of natural cement were imported into the United Kingdom, chiefly from Belgium, much of this inferior material having been sold as "Portland Cement," or as "Best Portland Cement," and imported in bags bearing only initials or marks instead of the maker's name, an ingenious device adopted for the purpose of avoiding the necessity, under the Merchandise Marks Act, of marking the packages "Made in Belgium."

Portland Cement.—This variety of cement is prepared by calcining an intimate mixture of limestone or chalk and clay at a temperature high enough to cause the raw materials to enter into perfect chemical combination.

The three essential components of the cement are lime, alumina, and silica; and these may be furnished either by pure or nearly pure limestone or chalk mixed with suitable clay obtained from any convenient source, or by rock of the kind employed in the manufacture of natural cement.

In either case, the proportions of the three components are carefully regulated so that the resulting cement shall comply with an established standard of chemical composition. The definite composition and uniform quality of the product are features clearly distinguishing Portland cement from the varieties of calcined calcareous rock known as natural cement.

Since the foundation of the modern cement industry in the early days of the nineteenth century, Portland cement has been greatly improved in quality by the strict regulation of the proportions of the raw materials, the perfect clinkering of the mixture, the fine grinding of the clinker, and by the adoption of scientific methods in every stage of manufacture.

In reinforced concrete work, the cement used should invariably be Portland cement of slow-setting quality, complying with the requirements of the British Standard Specification in respect of preparation, chemical composition, and physical properties.

The importance of slow-setting cement deserves to be

emphasized for the reason that, in practical construction, the operations of transporting newly mixed concrete to the points where it is to be used, and of depositing and tamping the material in moulds, should always be completed before the cement has commenced to set. It has been proved in practice that if concrete is worked after setting has commenced, its strength is seriously affected, the reason being that the process of crystallization in the cement is disturbed. The interlocking crystals are broken up, and the strength and cohesion of the material are proportionately reduced.

On reference to the British Standard Specification, it will be observed that the initial setting time of not less than 20 minutes, and the final setting time of not less than 2 hours nor more than 7 hours, are only intended for guidance in cases where a specially slow-setting cement is not required. For reinforced concrete work the initial setting time should be not less than 50 minutes, and the final setting time not less than 5 hours.

In some European countries the minimum time of initial set is specified at 60 minutes, and, referring to American cements, Professor Ira O. Baker says: "The quickest *Portlands* will begin to set in 20 to 40 minutes, but the majority will not begin to set under 60 to 90 minutes, and will not set hard under 5 to 6 hours."

Strength of Portland Cement.—The British Standard Specification states that the average breaking stress in tension of neat cement briquettes 7 days after gauging must be not less than 400 lb. per square inch, and that the average breaking stress of the briquettes 28 days after gauging must show an increase on the breaking stress at 7 days after gauging of not less than—

25 per cent.	when	the 7-day test	is above	400 lb.	and	not above	450 lb.
20	,,	,,	,,	450	,,	,,	500 ,,
15	,,	,,	,,	500	,,	,,	550 ,,
10	,,	,,	,,	550	,,	,,	600 ,,
5	,,	,,	,,	600	,,	,,	— —

Similarly, the average breaking stress in tension of 1 : 3 cement and sand briquettes, 7 days after gauging must be not less than 150 lb. per square inch, and 28 days after

gauging must be not less than 250 lb. per square inch, and the increase in the breaking stress from 7 to 28 days must be not less than—

25 per cent. when the 7-day test is above 200 lb. and not above 250 lb.
15 „ „ „ „ 250 „ „ 300 „
10 „ „ „ „ 300 „ „ 350 „
5 „ „ „ „ 350 „ „ ——

For comparison with these requirements, we give, in Table I, the results of a series of tensile tests conducted for the Reinforced Concrete Committee of the Institution of Civil Engineers on briquettes of "Ferrocrete" cement supplied by the Associated Portland Cement Manufacturers, Ltd.

TABLE I.

TENSILE STRENGTH OF PORTLAND CEMENT, IN POUNDS PER SQUARE INCH.

No. of Specimens Tested (each test).	Average Tensile Strength.			
	7 Days.		28 Days.	
	Neat Cement.	Cement and Sand 1 : 3.	Neat Cement.	C[illegible] and[illegible]
2	741	237	833	
1	817	296	893	
4	722	229	847	
2	637	212		
2	647	196		
6	617	196		
2	[illegible]	[illegible]		

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As cement can be t leaving as aggregate all particles of
than in compression The sand, if of suitable quality, can
adopted. It must b th the cement and aggregate in the
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material. Therefore th as possible in size, and carefully
alone does not afford proportion of voids to a minimum,
guidance as to the m anging from $\frac{1}{8}$ in. up to $\frac{3}{4}$ in. Any
to be employed in co ion must be broken to come within

Water.—Apart from other reasons, water is an essential constituent of concrete in the respects that it has the effect of commencing the series of reactions which result in the setting of the cement and that it enters into chemical combination with that material as water of crystallization. For this purpose alone, the proportion of water by weight should be at least 8 per cent. the weight of cement, but, in practice, a much greater proportion is requisite.

Fresh water should always be used, and care should be taken that the water is free from earthy, vegetable, and organic matter, acids, and alkaline substances in solution or suspension.

Tests by Alexandre, Feret, and others show that the use of sea water has very little effect upon the strength of concrete after the lapse of a year or more, the chief objection to its employment being that the setting and subsequent hard-ening of the concrete are retarded—a serious disadvantage ss constructive work.

In **Sand and Stone Screenings.**—It is obvious that no t is sptical objection can exist to the employment, as a mentstitute for sand, of small, well-graded particles of stone, rtlanar to the larger particles and fragments forming the majority wih Although the finer particles and coarse fragments will not set hard separately in practice, they should be graded *Strength of Portlan*g line actually exists

fication states that the as to the strength of mortar made of neat cement briquettes 7 days after gtively, attention must less than 400 lb. per square inch, and and grading of the breaking stress of the briquettes 28 daysials. Wherever stone show an increase on the breaking stree from dust and from gauging of not less than— in are objectionable in

25 per cent. when the 7-day test is above 400 l
20 " " " " 450 e should be composed of
15 " " " " 500 e from clay, chalk, lime,
10 " " " " 550
5 " " " " 600 er impurities. Mortars

Similarly, the average breaking strerespect of watertightness cement and sand briquettes, 7 days afges of clay, alum, and not less than 150 lb. per square inch,. On the other hand,

mortars of good quality, with properly graded sand, and an adequate amount of cement, must inevitably suffer in strength if cement is displaced by inert material, partly because the inert particles have no value in themselves, and partly because they break up the continuity of the interlocking crystals formed during the setting of the cement.

As previously stated, the sand, or small particles of stone, should be of varying sizes, from $\frac{1}{8}$ in. downwards. Fine sand and sand with grains of approximately uniform size are equally undesirable in practical work, as for given mixtures they produce mortars of less density and strength than those obtained by the use of coarser and well graded sands.

Aggregates.—The coarse material in concrete may be composed of gravel, shingle, or broken stone of any hard, close-grained, and durable character.

Among the materials which should never be used as aggregate in reinforced concrete work are: coal residues, including ashes, cinders, clinkers, coke breeze, pan breeze, and slag; blast furnace slag, copper slag, and forge breeze; sulphates and compounds of unstable character.

Limestones and other forms of calcium carbonate are inadmissible in concrete for buildings, owing to the fact that they are disintegrated at high temperatures, and broken brick is not to be recommended because it yields concrete of comparatively low compressive strength.

Aggregates should always be free from sand, earth, clay, quarry refuse, and other foreign matter. Where sandy gravel, or Thames ballast, is employed, the sand should be removed by screening, leaving as aggregate all particles of $\frac{1}{8}$ in. gauge and above. The sand, if of suitable quality, can afterwards be mixed with the cement and aggregate in the proportions which have been adopted for the preparation of the concrete.

The pebbles or fragments of broken stone used as aggregate should be varied as much as possible in size, and carefully graded so as to reduce the proportion of voids to a minimum, the sizes of the stones ranging from $\frac{1}{8}$ in. up to $\frac{3}{4}$ in. Any above the latter dimension must be broken to come within

the required limits, the reason for this being that particles of more than $\frac{3}{4}$ in. gauge are too large to pass readily between the reinforcing bars, or to be worked conveniently between them.

Proportioning Concrete.—In order that the constituents of concrete may be applied in the most advantageous manner it is necessary that careful attention should be devoted to the influence exerted by the characteristics of the aggregate and sand, the sizes of the particles, the proportions in which particles of various sizes are employed, the proportion of cement, and the consistency of the mixture.

As a general axiom, it may be stated that the density of concrete is the most important factor in respect of strength, impermeability, and economy.

The density of concrete is represented by the ratio of the volume of solid matter to the total volume of the concrete, and is the complement of the percentage of voids.

Maximum density is obtainable by the use of sand and aggregate of varying sizes, so that the gaps between the largest stones are occupied, as far as practicable, by stones of the next largest size, and thereafter by stones of progressively decreasing dimensions, until the voids remaining to be filled are so small as to give the sand or stone screenings the opportunity of continuing the process to a stage when the cement and water come into play and occupy all the interstices that remain.

Very little consideration is needed to show that if in a given volume of concrete the aggregate and sand are so graded and proportioned that the concrete is composed principally of stone the amount of cement required for the purpose of binding together the whole mass will be much less than the amount necessary in the case of concrete where the same care has not been taken to reduce the proportion of voids by proper grading and proportioning of the aggregate and sand.

While the employment of cement for the purpose of making up for the effects of badly proportioned aggregate and sand is merely waste of good material, cases often

arise where the proportion of cement can be increased with advantage, thereby producing concrete of proportionately greater strength.

The principles governing the proportions of concrete are covered by two well established laws—

1. For a given mixture of sand and aggregate, the strongest concrete is that containing the largest percentage of cement.

2. For a given percentage of cement and a given description of sand and aggregate, the strongest concrete is that obtained by such a combination of the fine and coarse particles as results in concrete of maximum density.

Various methods are adopted with the object of obtaining concrete of maximum density, some of these being briefly discussed in the succeeding paragraphs.

Proportioning by Voids.—This method is based upon the undeniably correct principle of determining the voids in the sand and the aggregate, using enough cement paste to fill the voids in the sand, and enough cement and sand so proportioned to fill the voids in the aggregate.

The voids in sand and aggregate may be determined approximately either by direct measurement or by computations based upon the specific gravity of the material and the weight of a unit volume of the particles constituting the sand or the aggregate.

The method of proportioning concrete by voids gives results that are only approximately correct. The percentage of voids in the aggregate varies with the compactness of the material as affected by different methods of handling, and the percentage of voids in the sand may be greatly affected by small variations of moisture. Moreover, it does not follow that the addition of cement paste equal to the voids as determined will actually fill the voids in the sand, nor that the addition of cement and sand equal to the voids as determined will actually fill the voids in the aggregate.

The cement paste covering the particles of sand virtually increases their size, and thereby tends to increase the voids, a similar tendency being due to the superficial tension of water in the cement paste.

So far as the aggregate is concerned, the actual volume of the voids does not correspond with the volume of sand and cement theoretically required to fill the voids, chiefly for the reasons that some of the larger particles of sand get between particles of aggregate that would otherwise come into close contact, and that, even in the coarsest aggregates some of the interstices are smaller in width than the finer particles of sand.

A general result is that, as previously stated, there must be an excess to the extent of about 10 per cent. of cement paste in the sand, and an excess of at least 10 per cent. of cement paste and sand in the aggregate, to provide for contingencies.

Proportioning by Trial Mixtures. —This method possesses the recommendations of simplicity and reliability, and can be usefully employed for the purpose of comparing the density of the concrete produced by different arbitrary mixtures, or mixtures prepared for the examination of their merits.

Briefly described, the method consists in placing a defined weight of each concrete mixture, including aggregate, sand, cement, and water, in a cylindrical vessel, tamping the concrete, and noting the height of the upper surface. The cylinder is then emptied, cleaned, and used in the same way for the trial of any required number of successive mixtures. The mixture of the greatest density will evidently be that giving the least height in the cylinder.

The cylindrical vessel can be formed by closing one end of a piece of iron or steel tubing, a convenient size being from 9 to 12 in. in diameter and from 12 to 15 in. in height. The tamping should be conducted as uniformly as possible, in order to minimize variations due to mechanical treatment.

Valuable as this method is as a ready means of determining the relative merits of various mixtures, its application would be impracticable for the purposes of an exhaustive investigation of the infinite number of mixtures obtainable by screening the sand and aggregate into various grades; and making combinations in different proportions.

Proportioning by Mechanical Analysis.—This scientific method of procedure is fully described and discussed in a lengthy paper, entitled "The Laws of Proportioning Concrete," by Mr. William B. Fuller and Mr. Sanford E. Thompson, in the *Transactions of the American Society of Civil Engineers*, Volume LIX, 1907.

Mechanical analysis consists in separating the particles of a sample of aggregate, sand, or cement—by means of a set of sieves, usually eight in number for each variety of material—into the various sizes of which it is composed, so that the results obtained can be plotted in a diagram to form a curve, each ordinate of which is the percentage of the weight of the total sample passing through a sieve having holes of a diameter represented by the distance of the ordinate from the origin in the diagram.

The objects of mechanical analysis curves, as applied to sand and aggregates for concrete, are—

1. To show graphically the sizes and relative sizes of the particles.

2. To indicate what sized particles are needed to make the mixture of sand and aggregate more nearly perfect, and so enable the engineer to improve it by the addition or substitution of other material.

3. To afford means for determining the best proportions of different sands and aggregates.

The experience of the authors of the paper mentioned is that the best mixture of cement, sand, and aggregate is one having a mechanical analysis curve resembling a parabola, which is a combination of a curve approaching an ellipse for the sand and a tangent straight line for the stone. The ellipse extends to a diameter one-tenth the diameter of the maximum size of stone, and beyond this point the stone is uniformly graded.

Mechanical analysis diagrams afford a very precise means of determining the proportions of any materials for concrete by sieving each of the materials, plotting the analyses, and combining the curves, so that the result is a curve as nearly similar as possible to the maximum density curve. The

proportions of the different materials required to produce the new curve will show the relative quantity of each which must be used in proportioning.

This method is particularly valuable, not only as a means of ascertaining the best proportions in which the sand and aggregate should be mixed, but also for the purpose of showing how the aggregate may be improved by increasing or decreasing the proportion of some particular size of stone. Mechanical analyses of the materials can be made from time to time during the progress of any work, so as to show whether or not the sizes of the particles of the aggregate have changed, and, in the case of any appreciable change, the proportions can be varied so as to produce the most economical concrete.

In order to secure the full benefit of this method of proportioning, the aggregate and sand must be screened to several sizes for combination in accordance with the curves representing the results of the mechanical analysis.

The apparatus necessary for a mechanical analysis consists of a set of sieves, preferably with a mechanically operated shaker, and scales for weighing. The number of sieves and the sizes of the openings in them depend upon the degree of accuracy required. For ordinary concrete work, a set of 8 or 10 sieves is sufficient.

Whether the increased cost of preparing materials in the manner described would be justified by the saving of cement is a question of somewhat problematical character. Even if the principle of mechanical analysis is not extended to the screening and grading of the aggregates employed in actual construction, the adoption of sieve analyses may be advantageous in works of considerable magnitude for the purpose of indicating improvements, sometimes attainable without additional cost, in the proportions of the materials.

Those desiring further information on the subject of mechanical analysis are referred to the paper already mentioned, and one on "Proportioning Concrete," by Mr. Sanford E. Thompson, in the *Journal of the Associated Engineering Societies*, Volume XXXVI, 1906.

A convincing proof of the value of the method is furnished

by the fact that it has been employed day after day by Mr. Fuller for determining the proportions for concrete used in constructing thin watertight walls, the proportions used having been about 1 : 3 : 7, whereas for watertight concrete where the materials are proportioned by other methods, 1 : 2 : 4 or richer mixtures are generally used.

Proportioning by Empirical Standards.—A method frequently adopted for settling the proportions of concrete to be used in different classes of work is to select one or other, of the mixtures stated or recommended in some handbook or code of regulations.

This method of procedure may be all very well in the case of ordinary building work or of construction where plain concrete of reasonable strength will satisfy requirements. In reinforced concrete work, where concrete of the highest quality is necessary, and where precise knowledge of the various properties of the material is of special importance, the method now under consideration is not to be recommended.

It cannot be too clearly stated that the adoption of a standard mixture does not necessarily result in the production of concrete of standard quality. Wide differences are caused by variations in the quality and fineness of the cement, the quality, sizes, and grading of the sand and aggregate, the methods of measuring the constituent materials, and the consistency of the mixture, even without taking account of differences due to variations in the important practical details of mixing, deposition, and ramming.

The dangers attending undue reliance upon empirical standards as a guide to the proportioning of concrete are very usefully brought out in a paper by Mr. T. J. Gueritte, M.Soc.Ing.Civ.(France).* The author shows the inadvisability of blindly adopting hard and fast specifications, without paying attention to the important question of the percentage of voids in the aggregate and in the sand. On the one hand, it tends to give a false feeling of security and leads to the belief that good concrete is bound to result

* "Concrete Mixtures for Ferro-Concrete Work," *Transactions of the Society of Engineers (Incorporated)*, Vol. IX, 1918.

from the use of clean and good materials in the proportions specified, whereas under certain conditions the concrete made in accordance with a generally approved formula, such as 1 : 2 : 4, for example, is dangerously porous. On the other hand, the quantities of materials required to make a cubic yard of concrete may vary within very wide limits, according to the nature of the aggregate selected.

Physical Properties of Concrete.—The strength and other physical properties of concrete are affected principally by the quality and quantity of cement; the quality, proportions, and grading of the sand and aggregate; the consistency, density, mixing, deposition, and tamping of the material, and the conditions under which the process of hardening or seasoning takes place.

Bearing in mind the numerous factors involved, it is evident that the strength of concrete must vary greatly, and that a definite standard of strength cannot be established even for concretes of nominally equal composition.

Average values might be stated with the object of showing approximately the strength which ought to be possessed by concretes of different kinds, but such a statement would not be of much use in actual practice.

So many tests have been conducted on concrete that the results would fill a very large number of pages if collected in a single volume. Many of the reports published are incomplete in the respect that the tests were undertaken chiefly with the object of determining the effect of varying some individual factor, all other factors remaining constant. Therefore, while useful for guidance as to the influence of the governing factors separately considered, results of the kind indicated are not suitable as a basis for general conclusions, and may lead to the formation of inaccurate opinions unless all attendant conditions are fully stated and duly appreciated.

Factors governing the Strength of Concrete.—With the foregoing points in view, the author has endeavoured, in the succeeding paragraphs, to denote the effects of various factors on the strength of concrete as shown by practical

experience and the results of well-authenticated tests,[1] but without reproducing the results of the latter in a form that might conduce to possible misconception.

Proportion of Cement.—As a general rule the compressive strength of concrete, other things being equal, may be regarded as varying with the proportion of cement per unit volume of the concrete. The variations shown by the Jerome Park tests, conducted by Messrs. W. B. Fuller and Sanford E. Thompson, may be expressed for different percentages of cement as follows, the strength of the concrete containing 8 per cent. of cement being taken at unity—

Percentage of cement	8	10	12½	15
Relative compressive strength at 140 days	1	1·25	1·57	1·9

In these tests the density of the concrete was practically constant, and the specimens of concrete were of similar quality.

Size and Quality of Aggregates.—Reliable tests show that with well-graded aggregates the density and strength of the concrete increase with the size of the largest particles, and the amount of cement required decreases as the size of the largest particles increases.

The advantages thus obtainable are practically confined to plain concrete work, as in reinforced concrete construction the largest particles of the aggregate must be comparatively small in order to enable them to pass between and around the bars and other members of the reinforcement.

Aggregates of low compressive strength, such as soft stone, brick, cinder, and some varieties of slag, necessarily produce weak concrete. The hardest and toughest qualities of rock yield concrete which is usually somewhat stronger than gravel concrete. In a general way, however, the difference is so small as to be practically negligible. Gravel concrete is frequently preferred for the reasons that the aggregate has a smaller percentage of voids than broken stone, that

[1] *Transactions of the American Society of Civil Engineers*, Vol. LIX.

the pebbles slide better into place owing to their rounded shape, and that the resulting concrete is generally superior in respect of impermeability.

The figures in the subjoined table are given as an approximately correct guide to the comparative strength of concretes made with different classes of aggregate, the strength of granite and trap concrete being taken at unity.

Description of Aggregate.	*Relative Strength.*
Granite and Trap	1·00
Gravel, Hard Limestone and Sandstone	0·90
Soft Limestone and Sandstone .	0·67
Cinders and other weak Aggregates	0·25

Consistency. So far as compressive strength, density, and impermeability are concerned, the best results are obtained by mixing concrete with sufficient water to make a mixture capable of flowing slowly in the moulds.

Dry mixtures develop greater strength than moderately wet mixtures at first, but the latter attain superior strength after the lapse of five or six months. In using dry mixtures it is difficult to secure uniform consistency and to obviate the risk of voids. On the other hand, in very wet mixtures the cement and sand are apt to flow away from the particles of aggregate, and the resulting concrete is always weaker than that given by dry or moderately wet mixtures. Another disadvantage attending the use of very wet concrete is that a proportion of the cement comes to the surface in the form of "laitance," having lost practically all its setting properties.

For reinforced concrete work, moderately wet mixtures are beneficial for the additional reason that, in flowing around the reinforcement, the material deposits a coating of cement upon the steel and effectively protects it from corrosion.

The relative compressive strengths of dry, moderately wet, and very wet concretes up to the age of six months

are represented by the curves in Fig. 10, which is based upon tests conducted at various laboratories for the American Concrete Institute.

Mixing.—The consistency of concrete is affected by efficient mixing as well as by the proportion of water used. The strength of concrete also depends materially upon the thoroughness of mixing.

Good concrete can be secured either by hand or by machine mixing, but in the former case the difficulty is to get the work done properly. The material must be turned over

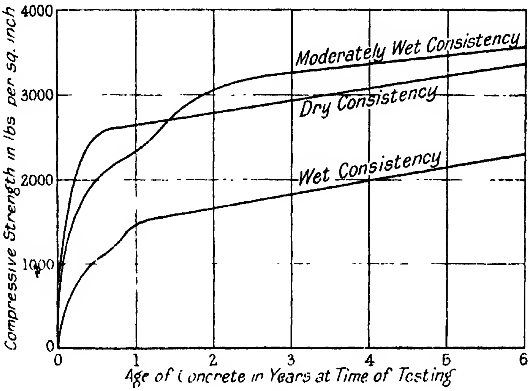


FIG. 10

thoroughly, the shovels must find their way to the boards, and the mixture must be picked up from the bottom and turned over and over again. This is hard work, and, especially in warm weather, the men are inclined to become optimistic, and to believe that a batch of concrete is finished when in reality it is only about half mixed.

Machine mixing is always to be preferred, for the reason that the work is more thoroughly performed and the resulting concrete is stronger than concrete of the same proportions mixed by hand.

Comparative tests conducted some years ago at the Watertown Arsenal, U.S.A., on hand and machine mixed

concretes showed an advantage of 11 per cent. in favour of machine mixing, the criterion being the compressive strength of test specimens made from the various batches of concrete. Another series of tests at the same place indicated an advantage of more than 25 per cent. in favour of machine mixed concrete. The results of similar tests at the University of Illinois on a large number of specimens showed that the average compressive strength of machine mixed concrete was nearly 28 per cent. greater than that of hand mixed concrete.

On large contracts, machine mixing is adopted as a matter of course, owing to its convenience and economy. On very small contracts, hand mixing is generally more economical. Under exceptional conditions, where space is not available for the installation and operation of mixing plant, the contractor has no choice in the matter and must perforce adopt hand mixing.

In order to compensate for the lower efficiency of hand mixing, the proportion of cement should be increased by from 10 to 15 per cent. in cases where this method is employed.

Treatment of Mixed Concrete.—Unless concrete is used before initial set has commenced and while the state of perfect admixture is maintained, the strength of the material may be seriously impaired. The reasons are that the re-working of concrete which has partially set has the effect of breaking up the crystals already formed and of reducing the interlocking effect resulting from the crystallization of the cement under normal conditions.

The strength of concrete is also affected by the manner in which the material is deposited and consolidated in the moulds, and by the conditions prevailing while it is in course of hardening. Concrete seasoned in air should be kept moist, especially in hot and dry weather, as moisture is conducive to the process of crystallization and therefore tends to promote increased strength. An excess of water is prejudicial whether in mixing or after deposition, because it gives rise to the formation of the inert substance known as "laitance." This is frequently liberated from concrete deposited under

water and wherever it is likely to occur, the proportion of cement should be increased by about 15 per cent, to compensate for the reduced strength that would otherwise result from the loss of cement by laitance.

Age of Concrete.—Numerous tests extending over long periods of time demonstrate the fact that concrete increases in strength with age. The rate of increase is rapid at the start, and gradually becomes slower. So far as can be

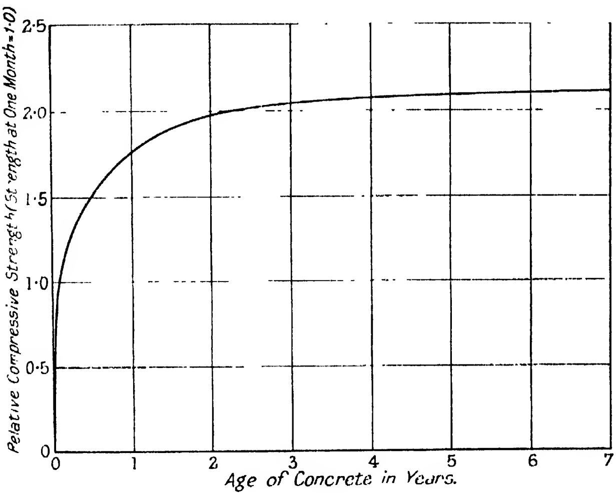


FIG. 11

inferred from experimental results, and the present condition of works executed long ago, there appears to be no reason why the increase of strength should not continue indefinitely.

As usual in the case of experimental records, the results of tests exhibit variations. Consequently, diagrams published in different treatises are not in precise agreement as to the rate of increase with age. For general guidance in practical construction, Fig. 11 and the subjoined table may be taken as a fair interpretation of the average results furnished by many series of tests in this country, in the United States, and on the Continent.

Age of Concrete.	*Relative Strength.*
1 month	1·00
6 months	1·50
1 year	1·78
2 years	1·98
3 years	2·00
7 years	2·10

In some cases, much higher rates of increase have been evidenced. For instance, six series of long period tests on concrete used in the construction of the General Post Office Extension, showed increases of strength over that at 54 days of 75 per cent. at six months; 100 per cent. at eighteen months; 135 per cent. at two years; 180 per cent. at three years; and 221 per cent. at five years.

Compressive Strength.—From what has been said in the preceding paragraphs, it must be evident that it would be impossible to formulate a reliable statement predicting the exact strength that will be possessed by concrete mixed in any given proportions. Experimental results vary between very wide limits, and it is not desirable that either the minimum, the mean, or the maximum value taken haphazard from some published record should be used as a basis for practical designs relating to reinforced concrete construction.

The engineer who wishes to combine safety with maximum economy must be in a position to estimate with a close approach to accuracy the effects of the various factors governing the strength of the concrete which he proposes to use. In some cases previous experience may be a sufficient guide, and in others it may be necessary to obtain additional data by testing specimens of the concrete.

For the purposes of the present work, it will be sufficient to state the conclusions of a few leading authorities concerning the minimum and maximum values for the compressive strength of concrete.

In their Second Report, the R.I.B.A. Committee wisely recommend that "before the detailed designs for an important work are prepared," specimens of the concrete should be tested and that "the average of the results should be taken

as the strength of the concrete for the purposes of calculation."

In the case of concrete made in proportions of 1 cement, 2 sand, 4 hard stone, the Report says that the compressive strength should not be less than 1,800 lb. per square inch at 28 days, and that such a concrete should develop a strength of 2,400 lb. per square inch at 90 days.

The London County Council Regulations for reinforced concrete buildings give minimum values for concrete, composed of cement, sand, and aggregate consisting of gravel or hard stone, and mixed in different proportions.

The proportions and minimum values are stated in the subjoined table—

PROPORTIONS BY VOLUME.			ULTIMATE COMPRESSIVE STRENGTH (MINIMUM).	
Cement.	*Sand.*	*Aggregate.*	*Age 1 month.*	*Age 4 months.*
			lb. per sq. in.	lb. per sq. in.
1	2	4	1,600	2,400
1·2	2	4	1,800	2,600
1·5	2	4	2,000	2,800
2	2	4	2,200	3,000

The Regulations include a formula for calculating the minimum compressive strength of concrete in intermediate proportions, but do not provide for the economic use of concrete possessing more than the minimum strength contemplated for each specified or any intermediate mixture.

The Report of the French Commission du Ciment Armé gives average values for gravel concrete, based on experimental results. The French values, expressed in British units and with the proportions stated in the manner adopted in this country, are given in the following table—

PROPORTIONS BY VOLUME.			AVERAGE COMPRESSIVE STRENGTH (MAXIMUM).	
Cement.	*Sand.*	*Aggregate.*	*Age 28 days.*	*Age 90 days.*
			lb. per sq. in.	lb. per sq. in.
1	1·75	3·5	1,520	2,280
1	1·56	3·12	1,706	2,550
1	1·37	2·75	1,890	2,850

The American Joint Committee on Concrete and Reinforced Concrete recommend that the maximum values tabulated below should be adopted for the ultimate compressive strength of concrete.

PROPORTIONS BY VOLUME.			ULTIMATE COMPRESSIVE STRENGTH (MAXIMUM).			
Cement.	*Sand.*	*Aggregate.*	*Granite or trap.*	*Gravel, or hard Limestone or Sandstone.*	*Soft Limestone or Sandstone.*	*Cinders.*
			lb. per. sq. in.	lb. per sq. in.	lb. per sq. in.	lb. per sq. in.
1	2	4	2,200	2,000	1,500	600
1	1·5	3	2,800	2,500	1,800	700
1	1	2	3,300	3,000	2,200	800

Comparing the minimum values, which are virtually the maximum values, of the London County Council with the maximum values of the American Joint Committee, we find considerably different estimates of ultimate compressive strength. Thus, for 1 : 2 : 4 and 1 : 1 : 2 mixtures, we have—

AUTHORITY.	PROPORTIONS. 1 : 2 : 4	1 : 1 : 2	AGGREGATE.
L.C.C. Regulations .	1,600	2,200	Granite, trap, gravel, or hard stone.
American Committee	2,000	3,000	Gravel or hard stone.
	2,200	3,300	Granite or trap.

The explanation of these wide differences is probably that the L.C.C. Regulations have been formulated with a special view to the establishment of safeguards against inferior work by builders with insufficient experience of reinforced concrete construction, while the recommendations of the American Committee are intended for the guidance of professional men and experienced contractors.

An admirable feature in the Report of the R.I.B.A. Committee is the recommendation that the actual strength of the concrete, as ascertained by tests, should be taken for

the purposes of calculation. This proviso is a direct encouragement to those who aim at the production of concrete of the highest possible quality, and the example of the Committee is one fully deserving imitation.

Concrete of considerably higher compressive strength than the maximum value recognized by any public authority has been produced for some years past by experienced engineers, and further advances have been made since the recent development of concrete shipbuilding.

A case in point is furnished by two series of tests made for H.M. Office of Works by Messrs. David Kirkaldy & Son, on concrete as used in building H.M. New Stationery Office, London. The concrete was moulded in the form of 6 in. cubes, which were tested at the age of 28 days. The results are summarized in the table below—

PROPORTIONS.	NO. OF TESTS.	AVERAGE COMPRESSIVE STRENGTH.
		lb. per sq. in.
1 : 1·6 : 3·2	18	3,658
1 : 1 : 2	16	4,840

Still higher results have been obtained from tests of special concrete for shipbuilding work. As an example we give below the average compressive strength of 6 in. cubes of concrete prepared under the direction of Major J. H. de W. Waller, D.S.O., R.E., at Lake Shipyard, Poole. The tests were conducted in May, 1918, by Messrs. D. Kirkaldy & Son.

PROPORTIONS.	AGE.	NO. OF TESTS.	AVERAGE COMPRESSIVE STRENGTH.
			lb. per sq. in.
$1\frac{1}{4} : 1 : 2\frac{1}{2}$	8 days	3	4,860
$1\frac{1}{4} : 1 : 2\frac{1}{2}$	28 „	3	7,070

Tensile Strength.— Numerous tests have been made from time to time with the object of determining the tensile strength of different qualities of concrete. But the number of such tests is small in comparison with that of other tests

of the same material, and the results are so disjointed that they afford little guidance to the engineer who desires to estimate the probable tensile strength of any particular kind of concrete he may have under consideration.

The statement has been made that the tensile strength of concrete is usually of little importance, because it is neglected in the design of reinforced concrete beams and kindred members. The tensile strength of concrete is, however, of much importance as it is directly connected with the resistance of reinforced concrete beams to diagonal tension. Consequently, the subject is one deserving much greater attention than it has so far received.

The tensile strength of concrete is generally regarded as being about one-tenth the compressive strength of the material. There is, however, no definite connection between the values in question.

The nature of the sand and aggregate appear to produce more effect upon the tensile than upon the compressive strength of concrete, and the same may be said of the proportions of the mixtures and the workmanship.

Owing to practical difficulties in the preparation and testing of specimens, the results hitherto obtained are generally lower than the true values, especially of the superior qualities of concrete produced of late, and it is to be hoped that more light will be thrown upon the subject by systematic investigation.

From the well-known tests of Professors Hatt, Henby, Talbot, Spofford, Woolson, and others it may be inferred that stone or gravel concrete in the proportions of 1 : 2 : 4 should develop a tensile strength of between 200 lb. and 300 lb. per square inch at the age of 28–30 days. These figures relate to the breaking stress of plain concrete, and it is important to bear in mind the fact that the ultimate failure of reinforced concrete in tension is appreciably retarded by the co-operation of the steel.

Transverse Strength.—As beams can be tested to destruction with less complicated appliances than those required for crushing tests, and as the results are more reliable than

those obtained by direct tension tests, the transverse strength of concrete is very frequently employed as an index to the tensile and the compressive strength of the material.

One of the most comprehensive series of transverse strength tests is one conducted by Mr. W. B. Fuller, at Little Falls, New Jersey, on a large number of 6 in. by 6 in by 72 in. beams, with 30 in. and 60 in. spans, made of neat cement, mortar, and concrete in proportions ranging down to 1 : 8 : 10. Full details are given in *Concrete, Plain and Reinforced*, by Thompson and Taylor. For our present purpose the following average results for concrete are quoted—

Proportions by Volume.	*Age when Tested.*	*Modulus of Rupture*
	days	lb. per sq. in.
1 : 1 : 2	33	710
1 : 2 : 3	"	471
1 : 2 : 4	"	139
1 : 3 : 6	"	226

According to Feret's experiments, the ratio of the modulus of rupture to tensile strength is practically constant at 1·95, and may be taken as ranging from 1·5 to 2 in general practice.

The ratio of the modulus of rupture to compressive strength usually ranges from $\frac{1}{4}$ to $\frac{1}{8}$, and appears to vary with the age owing to the more rapid growth of the compressive strength as compared with the tensile strength of concrete.

While the values derived from the results of transverse tests may be open to question, they are sufficiently accurate for comparisons and other purposes.

Shearing Strength.—Until comparatively recent years, the shearing strength of concrete was very generally stated at little more than one-tenth the compressive strength of the material. The reasons for the low value formerly accepted appears to have been either (1) that the methods adopted for the determination of shear permitted the failure of the test specimens by diagonal tension long before the ultimate strength in shear had been reached, or (2) that the term "shear" was used to denote complex action such as that

taking place in the web of a beam, where again diagonal tension is a governing factor. In either case the result would be the same, namely: the substitution of the tensile strength for the shearing strength of the concrete.

Improved methods of testing, which provide for elimination of the effects of diagonal tension and beam stresses generally, have demonstrated the fact that the strength of concrete in direct shear is actually from 60 per cent. to 80 per cent. of its strength in direct compression, and in some cases the percentage is still higher, as shown by the following table—

STRENGTH OF CONCRETE IN DIRECT SHEAR.

Proportions.	*Shearing Strength.*	*Compressive Strength.*	*Authority.*
	lb. per sq. in.	lb. per sq. in.	
1 : 2 : 4	1,480	2,345	Prof. Spofford
1 : 2 : 4	1,418	3,210	„ Talbot
1 : 3 : 6	1,150	1,410	„ Spofford
1 : 3 : 6	1,250	2,290	„ Talbot

These and other results are to be found in Bulletin No. 8 of the University of Illinois, 1906, and are sufficient to illustrate the point that shearing stresses, in themselves, can generally be resisted by concrete without the aid of reinforcement. The case is different, however, when shear is employed as a convenient measure of diagonal tension, for which careful provision must always be made.

Poisson's Ratio.—The ratio of deformation at right angles to the stress, to deformation in the direction of the stress in any material is termed Poisson's Ratio, here denoted by the symbol Π.

In the case of concrete the value of the ratio varies from $\Pi = 0{\cdot}5$ to $\Pi = 0{\cdot}25$. Professor Talbot states, in University of Illinois Bulletin, No. 20, 1908, that for 1 : 2 : 4 concrete he found the value of Poisson's ratio to vary between 0·10 and 0·17 up to about one-half the ultimate load. Beyond this point the value increased, the maximum being probably 0·25 at the ultimate load. For the purpose of calculations

where ordinary working stresses are involved, 0·10 is a fair average value for use.

Elastic Modulus.—The elastic modulus of concrete is represented by the quotient of the stress per unit area divided by the strain (either tensile or compressive) per unit length, and where pounds and inches are adopted as the units the modulus is expressed in pounds per square inch. The deformation of the concrete under load is measured by an extremely sensitive instrument, and the measurements for the different loads are plotted so that a curve is obtained showing the relation between stress and strain throughout the range of the test.

While the form of the curve is approximately parabolic, it follows what is practically a straight line until the stress has reached from 300 lb. to 400 lb. per square inch, and although the curve becomes greater beyond such a point, it is usually regarded as being replaceable by a straight line within the limits of working stress. This line is the basis of the "straight line" theory of reinforced concrete.

Values given for the elastic modulus of concrete range from 300,000 lb. to over 7,000,000 lb. per square inch, as determined by different investigators, the extent of the variations being due to the quality and age of the concrete and the basis upon which the modulus is calculated. There is also some difference between the values of the modulus for compression and tension, respectively.

Even if all existing data were of approximately uniform character, it would still be impossible to state in advance a precise value for the modulus of any given quality of concrete that might be used in practical work. Therefore the average value of 2,000,000 lb. per square inch generally recommended is one that can be safely adopted, especially as this corresponds fairly well with results obtained in numerous tests of beams, and as a considerable variation in the value of the modulus has comparatively little influence upon design.

Elastic Limit.—Although, strictly speaking, concrete has no elastic limit, there is a point in test diagrams beyond which a marked change in behaviour is noticeable, and this

point may be regarded as denoting the elastic limit for practical purposes. From experiments by Bach, van Ornum, and others, this limit may be placed at from one-half to two-thirds the ultimate strength of the concrete.

Coefficient of Expansion and Contraction.—As generally assumed, the coefficient of expansion for concrete approximates closely in value to that for steel, an important point borne out by the comparison of experimental results with the value of the coefficient for steel.

In three series of tests conducted by Professor Pence, at Purdue University, on 1 : 2 : 4 gravel and stone concrete, the average value of the coefficient was found to be 0·0000055 per degree Fahrenheit. The values ascertained at Worcester Polytechnic Institute, U.S.A., and at Columbia University, U.S.A., were 0·0000056 and 0·0000064 in one case, and 0·0000065 in the other.

To calculate the total change of length in a concrete structure in consequence of temperature variation, it is only necessary to multiply the coefficient by the product of the original length and the number of degrees Fahrenheit above or below the original reading of the thermometer.

Expansion and Contraction During and After Setting.—If the setting of concrete takes place in water, the concrete tends to expand, while on the other hand, if the process of setting is conducted in air, the concrete shows a decided tendency to shrink. In the latter case, internal stresses will be caused. Experimental data are not conclusive as to the change of volume that may be expected under either set of conditions. It is a fair inference that the expansion or contraction taking place must be approximately proportional to the amount of cement per unit volume, as the sand and aggregate cannot be affected to an appreciable extent.

In a paper contributed to the Western Society of Engineers, Professor A. H. White shows that concrete, even if a good many years old, will expand if immersed in water and will shrink if dried, and that such changes of volume are greater with rich than with poor concrete.

Resistance to Heat Conduction and Fire.—From a series of conductivity tests on gravel, stone, and cinder concretes, Professor Woolson, of Columbia University, drew the conclusions briefly summarized below.

1. All concretes have a very low thermal conductivity wherein lies their ability to resist fire.

2. When the surface of concrete has been exposed for hours to great heat, the temperature of the concrete 1 in. or less beneath the surface is several hundred degrees below that of the outside.

3. At a point 2 in. beneath a surface exposed to an outside temperature of 1,500° F. for two hours, the concrete will not be raised in temperature more than from 500° to 700°, and at points 3 in. or more beneath the surface, it will not be heated above the boiling point of water.

In another series of tests on the conductivity of steel bars embedded in concrete and projecting therefrom, Professor Woolson obtained practically identical results from cinder gravel and stone concretes. When the temperature of the bars and the concrete from which they projected was 1,700° F., the temperatures reached in the embedded steel at various distances from the heated surface were: 1,000° at 2 in., 400° to 500° at 5 in., and 212° at 8 in. His conclusion is to the effect that concrete is a very efficient protection to the embedded steel.

Other tests and the data furnished by numerous fires in all parts of the world clearly demonstrate the efficacy of reinforced concrete as a fire-resisting material. After prolonged exposure to severe heat, it is only to be expected that the surface may be damaged, but the concrete beneath and the imbedded steel are scarcely affected, and the construction as a whole retains its integrity, requiring little more than superficial repairs to make good the effects of fire and water.

As a general rule, the hardest and densest kinds of concrete are the best for withstanding heat, but cinder concrete, if of non-combustible character, offers considerable resistance to the transmission of heat owing to the low thermal

conductivity of the air confined in the pores of the aggregate. It must be remembered, however, that cinder concrete is unsuitable for any work where strength is an important factor.

Gravel and stone concretes may be relied upon for good results, but varieties of stone containing a large proportion of quartz are apt to split; and others, such as limestone, suffer disintegration when exposed to severe heat.

Watertightness.—In making concrete for use in combination with steel, it must always be remembered that watertightness, or impermeability, is often as important as strength. For ordinary construction, the quality of impermeability is frequently of minor importance. But where concrete has to protect the embedded steel from corrosion, and where reinforced concrete is employed in structures such as reservoirs, tanks, swimming baths, ships, barges, floating docks, caissons, conduits, underground chambers, and many others which need not be particularized, the greatest care must be taken to render the concrete capable of effective resistance to the passage of water.

There is really no serious difficulty in attaining this end, provided that the constituent materials are properly chosen, scientifically proportioned, and thoroughly mixed, and that the resulting concrete is carefully deposited and rammed.

Aggregate of non-porous character can easily be obtained, sand is non-porous and cement is non-porous after it has set. It is obvious, therefore, that water can only pass through concrete if any voids between the particles of the sand and aggregate remain unfilled by cement paste.

In scientifically proportioned concrete, the voids originally existing in the aggregate are filled as far as practicable by sand, and the voids originally existing in the sand are filled by cement paste, of which at least 10 per cent. in excess of the theoretical quantity should be used as a safeguard against possible inequalities of composition due to imperfect mixing or the partial separation of the ingredients during the interval existing between the delivery of the concrete

from the mixer and its deposition in the moulds. Again, if too much water is used, the cement mortar will tend to flow away from the particles of aggregate, forming porous "stone pockets," and if there is too little water porosity will also result. Therefore, a happy medium should always be the aim of the engineer.

However well the concrete may be proportioned, mixed and deposited, there may still be leakage through vertical, horizontal, or other joints occurring between portions of the work executed on successive days or at longer intervals, but by the exercise of proper care all such joints can be made impermeable.

Practical experience during the past quarter of a century shows that well-made concrete is practically impermeable, a fact demonstrated by the satisfactory behaviour of thousands of water reservoirs and tanks with thin walls. As an almost invariable rule, it has been found that any leaks existing in such structures occur at the joints, thus pointing to the impermeability of the concrete and to faulty work at the joints—a defect avoidable by the exercise of proper care.

The recent development of concrete shipbuilding in the United Kingdom has had the effect of directing further attention to the production of impermeable concrete, particularly in consequence of the necessarily strict requirements of the Admiralty and the Shipping Registry Societies.

The results of the laboratory tests conducted on concrete for shipbuilding purposes during the year 1918 constitute a most valuable series of records which it may be hoped will some day be published.

One point usefully emphasized by these tests is that the adoption of apparently satisfactory proportions of cement, sand, and aggregate will not ensure the impermeability of the resulting concrete in all cases. The fact is that a large, and even a lavish, proportion of cement will not suffice to make impermeable concrete unless the character and grading of the sand and aggregate are such that all voids are abundantly filled by cement paste.

TABLE II.

RESULTS OF PERCOLATION TESTS CONDUCTED BY MESSRS. DAVID KIRKALDY & SON ON CONCRETE SLABS, MEASURING 9″ × 9″ × 2″. WATER PRESSURE APPLIED OVER A SURFACE OF 5″ DIAMETER.

COMPOSITION OF CONCRETE: Aggregate 27 cub. ft., Sand 13½ cub. ft., Cement 8½ cwt., subject to variations with the percentage of voids.

No.	*Age of Slabs.*	*Kind of Aggregate.*	*Pressure per sq. in.*	*Duration of Test.*	*Remarks.*
1	20 days	Granite	30 lb.	4 days	Surface dry during 2 days; about ⅓rd of surface damp on 3rd day, but this became completely dry on 4th day. When broken, water had penetrated 1¾″.
2	20 days	Granite	30 lb.	4 days	About ¾ of surface damp after 1 day and water standing on surface on 3rd day. When broken, slab was completely saturated.
3	20 days	Whinstone	30 lb.	4 days	Surface covered with water after 1 day. When broken, slab was completely saturated.
4	28 days	(not stated)	30 lb.	4 days	Surface dry throughout test. When broken, water had penetrated 1¾″.
5	28 days	(not stated)	30 lb.	4 days	About ⅓rd of surface became damp in 2 hours, and about ½ surface damp in 4 hours. Slab then began to dry. On 2nd day, the surface was quite dry and remained so during 3rd and 4th days. When broken, water had penetrated 1¾″.
6	28 days	(not stated)	30 lb.	4 days	Surface was dry on 1st day, but about ½ became damp during 2nd day. Dampness gradually disappeared and surface was dry on 4th day. When broken, slab was just completely saturated.
7	41 days	(not stated)	30 lb.	4 days	Surface dry throughout test. When broken, water had penetrated ⅜″.
8	41 days	(not stated)	30 lb.	4 days	Surface dry throughout test. When broken, water had penetrated ⅛″ (½″ at one small spot).
9	41 days	(not stated)	30 lb.	4 days	Surface dry throughout test. When broken, water had penetrated $\frac{1}{16}$″ (⅜″ at one small spot).

A few typical results are given in Table II, the test specimens in every case being of concrete proportioned as follows—

Aggregate, a well graded mixture of stones from $\frac{5}{8}$ in. to $\frac{1}{4}$ in. gauge and free from sand .	27 cub. ft.
Sand, a well graded mixture of grains from $\frac{1}{8}$ in. gauge downwards	$13\frac{1}{2}$,, ,,
Cement	$8\frac{1}{2}$ cwt

These proportions to apply as long as the percentage of voids in the aggregate does not exceed 50 per cent. If that limit be exceeded, a corresponding quantity of sand and cement to be added.

Waterproofing Compounds.—Various foreign substances, such as hydrated lime, finely-pulverized clay, alum and soap in solution, and potash soap in solution, are much advocated by advertising firms for the purpose of securing watertight or "waterproof" concrete.

The following opinion as to the use of such substances is to be found in a report issued by the United States Bureau of Standards—

"The addition of so-called 'integral' waterproofing compounds will not compensate for lean mixtures, nor for poor materials, nor for poor workmanship in the fabrication of the concrete. Since in practice the inert integral compounds are added in such small quantities, they have very little or no effect on the permeability of the concrete. If the same care be taken in making the concrete impermeable without the addition of waterproofing materials, as is ordinarily taken when waterproofing materials are added, an impermeable concrete can be obtained."

Trowelled Surfaces.—The watertightness of concrete surfaces can be considerably increased by trowelling the material as it is deposited. Major J. H. de W. Waller, D.S.O., R.E., who has made use of trowelled plates of reinforced concrete in barge building, reports that samples of a plate 2 in. thick finished under working conditions were tested and found capable of withstanding a head of water of 480 ft. for two hours without leakage, and after the expiration of that period did not show actual leakage; what took place being best described by the

term "sweating." Under a head of 120 ft. water failed to penetrate another plate in five days, and when the plates were broken a penetration of only $\frac{1}{4}$ in. was shown by dampness.

Resistance to the Action of Sea Water.—It has been known for many years that chemical action takes place between the acids contained in sea water and the alkaline constituents of Portland cement, and that if concrete is permeable throughout its mass, the cement will gradually become disintegrated by the action of sea water.

Injury of the kind has happened and is still happening to many dock and harbour works, both in this country and abroad, but it is worthy of note that such destruction has been due to the permeability of the concrete, and that abundant proofs exist of the permanence of works constructed in impermeable concrete.

In an article by Mr. J. Watt Sandeman, M.Inst.C.E.,* it is stated that, "Permeability in concrete is due either to the porosity or the insufficiency of the mortar, or to faulty manipulation of the materials," and the writer expresses entire confidence in the reliability of well proportioned and properly applied concrete for resistance to disintegration by sea water.

M. R. Feret, the well-known French authority, says that the best means of protecting concrete from injury by sea water is to prevent the penetration of the water by making the concrete of maximum density. This condition is to be attained by the adoption of suitable proportions, a subject which has already been discussed in the present chapter. M. Feret emphasizes the importance of eliminating voids, and of making sure that the cement is not diluted by an excess of fine sand, which he considers "the greatest enemy of masonry in sea water."

An exhaustive investigation into the effects of sea water on concrete, conducted by Mr. R. J. Wig and Mr. L. R. Ferguson, led these well-known American engineers to the following conclusions—

1. All well-made Portland cements will resist disintegration if properly used.

* *Engineering*, 5th January, 1906.

2. The gauging of plain concrete with sea water will not be deleterious if proper methods of construction are followed.
3. Aggregates should be carefully selected, particularly with the idea of securing density and toughness.
4. Waterproofing compounds have no beneficial effect.

In work where concrete is deposited in sea water, special care must be taken to protect the concrete during the process of setting. Otherwise, the flow of water with the rise and fall of the tide may have the effect of washing the cement out of the green concrete, thereby leaving the material in a porous condition and very susceptible to disintegration by chemical action.

Although numerous marine structures composed of concrete prepared without special regard to the supreme importance of impermeability have undoubtedly suffered injury from the action of sea water, many hundreds of reinforced concrete structures have behaved in an entirely satisfactory manner, in all cases where the concrete has been correctly proportioned and applied.

Some long-period tests conducted in Boston Harbour, U.S.A., bring out very clearly the point that concrete of good quality is capable of effective resistance to injury by sea water, while concrete of inferior quality is seriously injured. In 1909, twenty-four reinforced concrete columns, 16 ft. long by 16 in. square were suspended from one of the piers in the Charlestown Navy Yard, by means of an iron ring at one end, about 18 in. being above the water at high tide and 4½ in. under water at low tide, thus providing for alternate immersion and exposure twice daily. After a period of five years had elapsed the columns were examined, and the observations made pointed unmistakably to the reliability of superior concrete. For example, columns Nos. 3 and 16, made of 1 : 1 : 2 concrete, were in splendid condition, while columns Nos. 7 and 8, of 1 : 3 : 6 concrete, had suffered considerable deterioration.

An interesting exhibit at the Museum for Nature Study and Technique, Munich, is a concrete block, one metre cube, which had been exposed for 50 years to sea water in a Danish

seaport. The Block is one of a number used in dock wall construction, and tests conducted by direction of the Danish War Minister are reported to have shown that the concrete was in excellent condition.

Action of Oils upon Concrete.—As a general rule, it may be taken that mineral, animal, and vegetable oils are not harmful to good concrete when it has hardened thoroughly.

Some vegetable oils, such as cocoanut oil and olive oil, which contain acids, have been found injurious, and the same effect has been observed in concrete structures where animal oils are heated to high temperatures.

Electrolysis in Concrete.—A good deal has been written concerning the possible injury of concrete, both plain and reinforced, by electrolytic action. Most of the fears which have been entertained respecting the deterioration of concrete and steel in consequence of electrolysis appear to be due to the mistaken assumption that the severe conditions established for the purpose of laboratory tests are likely to obtain in practical work. So far as the United Kingdom is concerned, the safeguards provided by the Board of Trade regulations are sufficient to obviate the risk of injury to reinforced concrete by stray electric currents. Negative evidence on this point may be found in the fact that the inquiries made by the Institution of Civil Engineers Committee on Reinforced Concrete failed to yield a single instance of electrolytic action in this country.

Even in the United States, where no restrictions exist with regard to stray electric currents, there is very little evidence of injury to reinforced concrete structures by electrolytic action. Referring to this subject in *Concrete, Plain and Reinforced*, Messrs. Taylor and Thompson say: "Injury to reinforced concrete from electrolysis is rare in practical construction, and much of the damage attributed to it has been due probably to other causes. Plain concrete, as shown by tests and experience, is never injured. The danger to structural steel, even if encased in concrete, is greater than to reinforced concrete."

CHAPTER III

FORMS AND PROPERTIES OF REINFORCING STEEL

Forms of Steel used as Reinforcement.—In general practice, the steel applied to the reinforcement of concrete is in the form of round bars or rods, thin strips corresponding to the familiar hoop-iron being frequently employed as web reinforcement in beams and analogous members.

Square and flat bars have been used occasionally, but are not to be recommended, as square bars cannot be obtained so readily or handled so conveniently as round bars, and flat bars are less efficient than those of circular form in respect of adhesion between the steel and the concrete.

Various special forms of bars have been devised, chiefly in the United States, with the object of providing a reliable mechanical bond in addition to the adhesion bond existing between the steel and the concrete. Some of these special bars are produced by twisting bars of square or of special cross sections, and others by methods of treatment which result in the formation of bars with corrugations, indentations, ribs, or other surface irregularities. One or two special types of reinforcement have been devised, consisting of bars with projecting wings intended to be turned up at any required angle so as to constitute web reinforcement for beams.

Many different types of network, including the well-known "expanded steel," have also been introduced for the reinforcement of slabs and other structural details wherein relatively small quantities of steel have to be distributed in an effective manner.

In ordinary practice, bars of merchant forms, readily obtainable from any steel manufacturer or merchant, are perfectly suitable, as the adhesion bond in conjunction with customary methods of anchorage are quite sufficient to guard against any slipping of the steel in the concrete.

Cases occur now and then where, owing to limitations of space, it may be difficult to provide for anchorage of the bars, and in any such circumstances an approved type of deformed bar may be employed with advantage. As a general rule, however, it may be fairly said that the chief advantage of special forms of bars rests with the patentee or maker, who is able to obtain a price much higher than that at which ordinary bars are obtainable in the open market.

Quality of Steel used as Reinforcement.—Some regulations governing building and engineering construction make the use of ordinary mild or structural steel obligatory in reinforced concrete work. In other cases, the designer is free to choose between mild steel and some form of high-tension or high-carbon steel.

As the elastic limit of mild steel averages about 35,000 lb. per square inch, and that of high-tension steel may be taken at about 50,000 lb. per square inch, some engineers have advocated the employment of high-tension steel with the object either of securing economy by the adoption of higher working stresses or of providing larger factors of safety.

Unfortunately, the elastic modulus for the two classes of steel is of substantially the same value. Consequently, under any given load up to the elastic limit, the deformation of high-tension steel is just as great as that of mild steel, and any increase of the working stress, beyond that considered safe for mild steel, must be attended by a corresponding increase in the width of the minute cracks, which always begin to appear on the tension side of a beam, or other member under transverse loading, long before the permissible working stress for mild steel has been reached.

Such cracks, even if increased in width by the higher stress adopted in the case of high-tension steel, do not involve structural danger if the reinforcement has been designed to take the whole of the tension developed, and in many forms of construction the cracks may be perfectly harmless when considered in connection with the protection of the steel from corrosive influences. In other forms of construction, such as water tanks, ships and marine works, and structures

in damp situations, it is certainly inadvisable to employ high-tension steel as a justification for increased working stresses. Therefore this variety of steel cannot always be adopted with economy, especially as in comparison with mild steel it is more costly and more troublesome of manipulation. The higher factor of safety obtainable is naturally an advantage, but this may be neutralized by the brittleness of the steel, a property characterizing material of inferior quality and making it necessary to obtain satisfactory test results before the acceptance of deliveries.

A variety of high-tension steel produced by some British and American patentees of deformed bars consists of mild steel, the yield point and ultimate strength of which have been artificially raised by mechanical treatment, a method fully discussed by Dr. W. C. Unwin in his classical treatise, *The Testing of Materials of Construction*, and to which the reader is referred for detailed information.

Steel so treated possesses all the advantages and none of the disadvantages of high-carbon steel, and it has been found by tests that bars which have been thus physically developed by twisting after having been rolled, do not show the amounts of permanent set observed in ordinary bars when tested for the first time, and which are probably due to initial lack of straightness of the bars or to small defects of homogeneity in the material.

Although in no way detrimental in steel construction, these preliminary deformations in ordinary steel bars are to a small extent disadvantageous in reinforced concrete work, for the reason that they cause unnecessary strains in the concrete, in addition to the strains due to the subsequent elastic deformation of the steel. Consequently, in a reinforced concrete beam, hair cracks on the tension side are developed at a somewhat earlier stage and are somewhat wider than they would be if the steel were not liable to preliminary deformation. Thus, the elimination of initial permanent set from physically developed bars is clearly a desideratum, even though its influence may not be of great magnitude.

A point to be noted is that the effect produced by the mechanical treatment of mild steel is entirely removed by annealing, and it follows that the high tensile properties artificially developed in the manner described may disappear if the steel is subsequently exposed to conditions similar to those established in the process of annealing.

Tensile Strength.—In the British Standard Specification for Structural Steel for Bridges, etc., and General Building Construction, it is stated that the tensile breaking stress of round and square bars shall be "between the limits of 28 and 33 tons per square inch of section, with an elongation of not less than 20 per cent. measured on the Standard Test Piece B," this test piece having a gauge length not less than eight times the diameter.

Therefore, employing the units customary in reinforced concrete literature, the tensile strength of mild or structural steel should vary between 62,720 lb. and 73,920 lb. per square inch.

The tensile breaking strength of high-tension steel varies with the percentage of carbon, a point illustrated by Table III, where the breaking strength is given for different qualities of steel ranging from very mild low carbon steel to hard high carbon steel.

TABLE III.

TENSILE BREAKING STRENGTH OF STEEL WITH DIFFERENT PERCENTAGES OF CARBON. (*Bauschinger.*)

Percentage of Carbon.	*Tensile Breaking Strength.*
	lb. per sq. in.
0·14	62,944
0·19	68,096
0·46	75,712
0·51	79,744
0·54	79,072
0·55	80,416
0·57	79,744
0·66	89,600
0·78	92,064
0·80	102,816
0·87	104,608
0·96	118,048

The percentage of carbon in steel for various purposes is stated below as an index to the percentages of carbon stated in Table III.—

	Percentage of carbon.
Boiler Plate	From 0·10 to 0·33
Wire Ropes (ordinary)	„ 0·15 to 0·30
„ (hard)	„ 0·45 to 0·75
Structural Sections	„ 0·20 to 0·25
Rails	„ 0·30 to 0·60
Tool Steel.	„ 0·70 to 1·40

With regard to high-tension steel produced by the mechanical treatment of mild steel bars, it is impossible to state precise values for general application, because the amount of work put on a bar appreciably affects its tensile strength and other properties.

From a series of tests conducted by Messrs. D. Kirkaldy & Son on mild steel bars twisted after rolling, it appears that the ultimate tensile strength was increased from about 70,000 lb. to 86,000 lb. per square inch, nearly 23 per cent.

Compressive Strength.– For all practical purposes, it may be assumed that the strength of steel in compression is equal to that of the same quality of metal in tension.

Shearing Strength.—The resistance of steel to shearing stress varies somewhat, according to the direction in which force is exerted with respect to the structure developed in rolling the plate or bar.

As a general rule, however, it may be taken that the ultimate shearing stress of steel is from 70 to 75 per cent. of the ultimate tensile stress. For mild steel the value is about 50,000 lb. per square inch, and for hard steel about 80,000 lb. per square inch.

Ductility.— Useful evidence as to the ductility of steel bars may be obtained by the cold bending tests prescribed by the specifications of the Engineering Standards Association. Steel of high elastic limit, whether due to a large percentage of carbon or to mechanical treatment after rolling, should on no account be accepted unless satisfactory proof is forthcoming as to its ductility.

Elastic Limit and Yield Point.—The following definitions are those given in Report No. 56 by the Engineering Standards Association—

"*Elastic Limit.*—The elastic limit is the point at which the extensions cease to be proportional to the loads. In a stress-strain diagram plotted to a large scale it is the point where the diagram ceases to be a straight line and becomes curved."

"*Yield Point.*—The yield point is the point where the extension of the bar increases without increase of load."

In the case of steel, it may be assumed that the permanent deformations occurring up to the elastic limit are small enough to be neglected, and that the elastic deformations conform to Hooke's Law that the strain is proportional to the stress. As this proportionately diminishes gradually after the elastic limit has been exceeded, and very rapidly when the yield point has been reached, it is evident that the elastic limit is of more importance than the yield point to the reinforced concrete designer. In fact, the elastic limit denotes a point up to which reinforced concrete is able to recover without injury from the effects of stresses far in excess of safe working stresses.

The elastic limit of steel is not quite the same in tension and in compression, and although not bearing a fixed relation to the ultimate strength of the material, its average value may be taken at from 50 to 60 per cent. of the ultimate strength both in tension and in compression.

On this basis, the elastic limit of mild or structural steel is between 31,000 lb. and 44,000 lb. per square inch.

The series of tests by Messrs. D. Kirkaldy & Son, mentioned in a preceding paragraph, showed that for mechanically treated steel bars with an ultimate tensile strength of 86,000 lb. per square inch, the yield point was 69,000 lb. per square inch, or fully 80 per cent. of the ultimate strength. It does not follow, however, that a similarly high rate of increase will always be secured by mechanical treatment.

Elastic Modulus.—Numerous well authenticated tests show conclusively that the elastic modulus, alternatively

termed the *modulus of elasticity* and the *coefficient of elasticity*, of steel is practically constant at 30,000,000 lb. per square inch within the elastic limit, this average value applying to the modulus in tension, compression, and bending, and for all descriptions of steel.

Elastic Extensibility and Compressibility.— The extent to which steel will be stretched or compressed under load can be computed with sufficient accuracy by the aid of the above-mentioned value of the elastic modulus.

Considering three kinds of steel under tensile stresses ranging from 10,000 lb. to 20,000 lb. per square inch, and also under stresses corresponding with the average elastic limits of the three varieties of steel, the elastic extensions per unit length of 1 in. will be as given in the following table—

TABLE IV.

ELASTIC EXTENSION OF STEEL.

Kind of Steel	*Elastic Limit.* lb. per sq. in.	*Tensile Stresses in pounds per square inch.*					
		10,000	15,000	20,000	34,000	50,000	65,000
Mild (low carbon) . .	34,000	0·00033	0·00050	0·00066	0·00113	—	—
Hard (high carbon) . .	50,000	"	"	"	"	0·0016	—
Mild (low carbon) Mechanically treated.	65,000	"	"	"	"	"	0·00216

The figures in this table may be useful as a clear reminder of the fact that however hard and strong a steel may be, and however high the elastic limit, yield point, and ultimate strength, it will inevitably be extended or compressed under any given load up to the elastic limit, just as much as the softest and weakest variety of steel.

Coefficient of Expansion and Contraction.— The value of this coefficient for steel is variously stated at between 0·0000056 and 0·000007 per degree Fahrenheit and is usually taken at 0·0000065 for mild steel and 0·000007 for hard steel.

CHAPTER IV

PROPERTIES OF CONCRETE AND STEEL IN COMBINATION

The general characteristics and distinctive structural features of reinforced concrete have been stated in Chapter I, and a discussion of the general theory of reinforced concrete construction will be found in Chapter V. The scope of the present section is limited to the properties of concrete and steel in combination.

Adhesion Between Concrete and Steel Bars.— Adhesion of the concrete to the steel, in spite of stresses caused by the load or by variations of temperature, is a matter of prime importance, and without it the distinctive qualities of reinforced concrete could not be realized. The high resistance of the combination is largely due to the fact that the constituent parts adhere so strongly one to the other that its action is akin to that of a homogeneous structure. Adhesion enables the concrete to offer resistance against sliding along the surface of the steel, and thereby facilitates the transference of forces from one material to the other. The result is uniformity of action on the part of the two materials. Should the resistance to sliding fall below the required amount, the materials will act independently, and the resistance of the structure will be greatly reduced.

Numerous experiments have been carried out with the object of determining the adhesion bond between concrete and steel.

The investigations conducted at various dates by Bauschinger, Feret, Hatt, Ritter, Talbot, Withey and others show that the value of the adhesion bond varies from about 200 lb. to nearly 900 lb. per square inch of surface contact.

Some interesting results are recorded in the Report of the French Commission du Ciment Armé. The tests were made upon an old beam of reinforced concrete in which $\frac{3}{16}$ in. rods used as reinforcement developed an adhesion bond of

from 1,138 to 1,308 lb. per square inch of contact surface. These high results are attributable in part to the crookedness of the rods, and are of some practical value, since the rods and bars employed as reinforcement are never absolutely straight.

The efficiency of the bond varies greatly with the surface condition of the bars, the presence or absence of rust, the quality, consistency, and shrinkage grip of the concrete, and the age of the bond.

Ordinary round steel bars are nearly three times as effective as smooth cold rolled or turned bars.

Bars with a thin film of rust give a bond about 15 per cent. higher than cleaned bars.

From tests at the University of Illinois and the University of Wisconsin it appears that concrete in the proportions of 1 : 2 : 4 gives a bond averaging 25 to 50 per cent. higher than 1 : 3 : 6 concrete, and a corresponding advantage is given by richer mixtures, the superiority of concrete mixed with an ample amount of water being very marked in respect of bond resistance.

The effect of age is shown by the tests of Feret, who found the bond resistance at an age of two years was approximately 50 per cent. higher than at an age of three months.

Tests conducted by Professor Talbot at the University of Illinois showed that when the loads applied were sufficient to cause slipping of the bars in the concrete, the frictional resistance to movement was still considerable. This resistance, taken when the bars had slipped about $\frac{1}{4}$ in., amounted to from 54 to 72 per cent. of the bond developed in the case of mild steel bars, and to from 32 to 49 per cent. in the case of cold rolled shafting.

An examination of the various records available leads to the conclusion that for ordinary plain steel bars, the bond resistance may be safely taken at from 350 lb. to 450 lb. per square inch of contact surface.

Effect of Deformed Bars giving Mechanical Bond.—With the object of providing a mechanical bond in addition to the adhesion bond between concrete and steel, many forms

of indented, corrugated, twisted, and otherwise deformed bars have been placed upon the market.

The efficiency of such bars depends mainly upon the grip and shearing strength of the concrete. Types of deformed bars with lateral projections tend to split the concrete, as demonstrated by the experiments of Professor Talbot at Illinois University and Professor Bach at Stuttgart. Such bars cannot be drawn through the concrete without shearing off an area equal to the area of the lateral projections.

The value of the mechanical bond given by deformed bars naturally depends very much on the nature and extent of the projections or other irregularities presented by the surface, but for general guidance it may be taken that the bond is from 50 to 60 per cent. greater than that given by plain steel bars.

A point worthy of notice is that in bars having lateral projections a considerable proportion of the metal becomes ineffective as reinforcement for the reason that tensile and compressive stresses cannot deviate so as to bring into play the metal employed in the form of projections.

Modular Ratio for Concrete and Steel.—Assuming the maintenance of the adhesion bond, any stress developed in a reinforced concrete member or structure must strain or deform the concrete and the steel to the same extent.

The stresses actually developed in the two materials by the application of any load are proportional to their elastic moduli, being governed by the modular ratio $m = Es/Ec$.

If we take $Es = 30{,}000{,}000$ lb. per square inch, and $Ec = 2{,}000{,}000$ lb. per square inch, then $m = 15$, a value now almost universally accepted as a satisfactory basis for practical purposes.

As the elastic modulus (E) for any material is the reciprocal of the extension or compression per unit stress and unit length, it represents the force that would, if such a thing were possible, stretch a bar of any material to double the original length, or compress it to an equivalent extent.

Thus, the statement that the value of Es for steel is 30,000,000 lb. per square inch implies that a force of 1 lb.

would stretch or compress a bar of 1 in. area by $\frac{1}{30000000}$, or $0\cdot00000003\dot{3}$ of its original length. Similarly, if the value of *Ec* for concrete is 2,000,000 lb. per square inch, a force of 1 lb. corresponds with an extension or compression of $\frac{1}{2000000}$, or 0·0000005 of the original length of the bar. Therefore the application of tensile force of any given intensity within permissible limits to a unit area of concrete will be followed by an extension fifteen times the length of the extension caused by the application of an equal force to a unit area of steel. Hence, for an equal elongation, the steel will carry fifteen times the stress carried by the concrete. The conditions are naturally precisely similar in the case of compression.

Assuming the stress to be taken by the concrete in compression is limited to 600 lb. per square inch, and that the concrete is to work in unison with the steel on the basis $m = 15$, the stress in the steel cannot exceed $600 \times 15 = 9{,}000$ lb. per square inch.

On the other hand, if we start with a stress of 16,000 lb. per square inch for the steel, the stress in the concrete will be 1,066 lb. per square inch. This may be safe for good concrete in compression, but is far beyond the ultimate strength of the material in tension, a matter which is discussed in the next paragraph.

Thus, the value $m = 15$ imposes a serious limitation on the employment of steel as compression reinforcement. If the modulus *Es* could be increased sufficiently, or the modulus *Ec* could be decreased sufficiently, to make $m = 30$, reinforced concrete designers would enjoy far more latitude than they now possess. In that event, starting with a permissible compressive stress of 600 lb. per square inch in the concrete, we should get $600 \times 30 = 18{,}000$ lb. per square inch in the steel; or, conversely, starting with 18,000 lb. per square inch in the steel, we should have only 600 lb. per square inch in the concrete. If concrete were used similarly in tension, its ultimate tensile strength would be far exceeded.

In former times, the value $m = 40$ was in common use, but modern investigations have shown $m = 15$ to be a

correct approximation for concrete of average quality as used in the present day, and there is no reason for anticipating any material change in this value.

Tenacity and Extensibility.— A good many years ago the remarkable elastic strength displayed by reinforced concrete structures and the apparent absence of cracks in members subjected to loads causing considerable deflection, led to the belief that, when working in combination with steel, concrete could be stretched to a far greater extent, without causing actual rupture, than would be possible in the case of plain concrete.

This attractive theory was supported by Considère, who, in a report to the Académie des Sciences, in 1902, claimed that his tests of reinforced concrete beams showed the ultimate extensibility of the concrete to be from ten to twenty times the extensibility of plain concrete tested in a similar manner.

The conclusions of Considère aroused much interest and were subsequently disproved by careful tests conducted by Professors Bach and Kleinlogel on the Continent, and Professor Turneaure in the United States.

As Professor Turneaure justly remarks: "In experiments of this sort it is extremely difficult to determine just when the concrete begins to crack. The steel forces it to elongate practically uniformly, even after rupture begins, so that a crack will open up very slowly and will, therefore, remain almost invisible for some time."

A very delicate method of detecting incipient cracks was accidentally discovered at the University of Wisconsin about 1902. It was then observed, in testing beams seasoned in water and only partially dried, that the appearance of a very fine hair crack was preceded by a dark wet line across the beam. Subsequent investigations, the results of which were published in 1906, showed that the "water-marks" denoting incipient cracks occurred at practically the same deformation at which plain concrete was ruptured.

Proof of the fact that the water-marks actually indicated the presence of incipient cracks was obtained by sawing out

and examining portions of the concrete showing marks of the kind.

Further investigation by Professor Bach, at Stuttgart, led to the conclusion that the water-marks appear at places where adhesion between the particles of concrete is diminished just before the formation of cracks. In plain concrete, every water-mark develops into a hair crack, whereas in reinforced concrete the steel has the effect of retarding the development of cracks at some of the water-marks, and of entirely preventing this formation at others.

The contrast between the behaviour of plain and reinforced concrete is made clear by the facts that in plain concrete, failure occurs suddenly and is accompanied or preceded by the opening of one or more large cracks, while in reinforced concrete, failure takes place gradually and only after ample warning, numerous small cracks developing simultaneously in such a manner that the total elongation at final rupture is far greater than that of plain concrete.

Thus, by the employment of steel the extensibility of concrete may be developed to the degree stated by Considère, whereas the extensibility of plain concrete is limited to the amount possible at the weakest cross section.

As incipient cracks first appear in reinforced concrete at an elongation corresponding to a tensile stress in the steel of about 5,000 lb. per square inch, it is evident that no allowance should be made for the tensile resistance of the concrete in practical designs, but investigation has shown conclusively that the minute cracks occurring under ordinary working stresses, do not expose the reinforcement to corrosive influences.

Expansion and Contraction.—As the coefficients of expansion of concrete and steel are of nearly equal value, very little stress in either material will result from variations of temperature, particularly in view of the effective nature of the insulation provided by the concrete in which the steel is embedded.

The expansion and the contraction of concrete in the processes of setting and hardening necessarily give rise to

internal stresses if the adhesion bond remains perfect. The expansion of water-hardened concrete tends to stretch the embedded steel, and the reaction of the steel tends to restrain the expansion of the concrete, the result being the development of tension in the steel and compression in the concrete. On the other hand, the contraction of air-hardened concrete tends to compress the embedded steel, and the reaction of the steel tends to restrain the contraction of the concrete, the result being the development of compression in the steel and tension in the concrete.

Experiments by Considère showed the contraction of 1 : 3 air hardened mortar, reinforced with 5·5 per cent. of steel to be only 0·01 per cent., and as sand and aggregate cannot be affected the contraction of concrete must be considerably less than that of mortar. Moreover, it must be borne in mind that the gradual hardening of the concrete is probably accompanied by molecular adjustment having the effect of obviating, or of largely reducing, internal stresses in reinforced concrete due to expansion or contraction in setting and hardening.

In reinforced concrete structures restrained by exterior forces which are more rigid than the reinforcing steel and so interfere with expansion and contraction, the stresses in the concrete are likely to be high, and the tensile stress developed may exceed the ultimate strength of the material. In such cases, the effect of the reinforcement will be to prevent the formation of large cracks, and to develop the extensibility of the concrete by providing for the necessary elongation by means of a well-distributed succession of minute cracks, invisible to the naked eye and of no practical importance.

Preservation of Steel Embedded in Concrete.—Steel embedded in concrete of proper consistency is most efficiently protected from corrosion by the film of cement which attaches itself to the surface of the metal; or, if the metal is rusty at the time of use, by an impervious coat formed as the result of chemical action between the metallic oxide and the cement.

Numerous instances are recorded of iron having been

embedded in mortar and concrete for long periods without suffering corrosion. A few of these are given below.

Iron clamps laid in mortar joints in the Parthenon have been uncovered in modern times and found to be in good condition after 2,000 years. Other examples have been cited showing the perfect condition of iron after having been embedded in lime mortar for upwards of 500 years.

At the Engineering Conference held at the Institution of Civil Engineers in 1907, Mr. W. T. Douglass, M.Inst.C.E., said that "in 1881, when taking down the Eddystone Lighthouse, built by Smeaton in 1757, he discovered a small bundle of iron rods which had been embedded by accident in Aberthaw lime concrete," and that "the colour of these rods was just as if they had come from the mill, and there was no mark of rust on them whatever."

Speaking at the same meeting, Mr. J. Hannay Thompson, M.Inst.C.E., said that having recovered the heads of several reinforced concrete piles which had been cut off after having been subjected to very heavy driving and left in the water of Dundee Harbour for about three years, the steel was found to be perfectly blue when the concrete was stripped off. An experiment by the same engineer to test the effect of sea water on reinforced concrete was carried out by making two blocks of concrete, 5 ft. long, inserting in each of them two clean steel rods and two very rusty bolts, and leaving the blocks in sea water for three years. On breaking the blocks open, the new steel rods were still quite blue, and the rust had disappeared from the old bolts.

A practical test extending over eight years was undergone by several reinforced concrete pile heads cut off during the construction of a wharf at Southampton in 1898, and allowed to remain on the shore, where they were alternately covered and exposed by the flow and ebb of the tides which occur there four times daily. Some of these pile heads were sent eight years or more later to certain railway and harbour engineers who desired to investigate the condition of the reinforcing bars.

Referring to two of these pile heads at the Engineering

Conference, 1907, Mr. C. S. Meik, M.Inst.C.E., said of one: "The exposed steel work on this specimen was much corroded, whereas the bars in the body of the concrete, on being cut open, were found to be quite free from any rust and as fresh as the day they were put into the pile." Mr. Cuthbert A. Brereton, M.Inst.C.E., said of the other: "The steel and iron work in the centre showed no deterioration whatever. In fact, the blue scale was still on the rods just as at the time they were put in."

The author may add that he had previously examined a similar pile head from Southampton and found the embedded steel in perfect condition.

A question sometimes raised is whether the steel in reinforced concrete may not be corroded in cases where minute hair cracks exist in concrete, either in the open air or under water. The point is obviously one of much importance to engineers employing reinforced concrete in marine work, and particularly in ships and floating structures. Careful tests conducted in 1907 at the Royal Department for Testing Materials, Gross Lichtenfelde, show conclusively that no danger of corrosion exists if the concrete is of the quality and consistency adopted in average practice, and if the steel is not stressed beyond the elastic limit. The test specimens employed were subjected while under load to the action of a mixture of oxygen, carbon dioxide, and water vapour for periods ranging between three and twelve days, and the steel was in every case found to be free from rust even when stressed up to the elastic limit. On the other hand, unprotected steel was seriously corroded after an exposure of two hours.

In this connection, it must be noted that steel which is slightly rusted before being embedded in concrete is protected by the insoluble and impermeable chemical compound formed by the combination of the rust with the alkaline constituents of the cement.

CHAPTER V

THEORY OF REINFORCED CONCRETE
INTRODUCTORY NOTES: FLEXURE OF BEAMS

INTRODUCTORY NOTES

Varieties of Structural Members.—Three classes of structural members are considered in this chapter: (1) beams, (2) compression members, and (3) tension members. As most frequently employed, these members are subject to simple bending, simple compression and simple tension, respectively. But bending moment is often accompanied by either compression or tension, giving rise to the combined stresses of bending and compression, or of bending and tension.

In reinforced concrete work, beams, in one form or other, are very widely used; compression members, although representing an application of reinforced concrete of considerably less economy than that given by beams, are extensively employed, and tension members, for which reinforced concrete is not at all suitable, are used very seldom. Members subject to combined stresses may be regarded either as beams, compression members, or tension members, according to the duty for which they are primarily designed.

Definitions.—For the purposes of this chapter the following definitions are given.

Beam, a term covering beams, bressummers, cantilevers, girders, lintels, slabs, and other members subject to flexural stresses.

Compression Member, a term including columns, piers, piles, pillars, posts, props, stanchions, struts, and all members, whether vertical or at any angle, subject to longitudinal compression applied more or less axially.

Tension Member, a term applying to all forms of ties subject to longitudinal tension applied axially.

It should be noted that combined stresses will be developed

if the force acting upon a compression or a tension member is non-axial.

Fundamental Principles.—The distinctive characteristics and properties of reinforced concrete are discussed in Chapters I and IV. From the statements there made, it is evident that the material is one combining in a remarkable way the most valuable properties of concrete and steel.

The compressive strength of the concrete is advantageously developed and the impermeability and durability of the material are usefully employed in the protection and preservation of the steel.

At the same time, full advantage is taken of the tensile strength of the steel, and in many cases a similar advantage is also taken of its compressive strength, in such a way that the extensibility of the concrete is greatly developed, or that both its extensibility and its compressibility are greatly developed. Although the elastic properties of the concrete are not actually changed in virtue of its combination with steel, as was once believed, the material really behaves very much as it would if such a change had taken place. Instead of being, like plain concrete, brittle and liable to sudden rupture under relatively small tensile stresses, reinforced concrete possesses toughness and elasticity and always gives ample warning of impending rupture, which only takes place after the elastic limit of the steel has been exceeded.

Therefore, the fundamental principles of reinforced concrete embody the combination of the constituent materials in a manner providing for the resistance of compression by the concrete and the resistance of tension by the steel; and they also cover the much less economical, but convenient and frequently very useful, supplementary combination of the materials wherein the concrete and the steel work together in resisting compression.

Relative Stress Intensities in Concrete and Steel.—In discussing the theory of reinforced concrete, it is here assumed that the adhesion bond between the concrete and the steel is perfect, and therefore that the deformation of the two materials is of equal amount. Reinforced concrete

construction is generally dependent upon this equality of action, and the results of numerous tests show that, from the practical point of view, it occurs in all structures which have been correctly designed.

As the elastic modulus represents the ratio of stress to strain, it is obvious that for equal deformations the strains in the two materials will be equal and the stresses in the concrete and the steel must vary as the elastic moduli of the two materials.

Thus, if t = tensile stress in the steel, tc = tensile stress in the concrete, Es = elastic modulus of the steel, and Ec = elastic modulus of the concrete, the following must be the stress relations: $t/tc = Es/Ec$. Hence we see the importance of $m = Es/Ec$, the modular ratio.

FLEXURE OF BEAMS

Stress Distribution in Beams.—As an introduction to the study of reinforced concrete beams, the nature of the stresses developed by bending moments in a plain concrete or other homogeneous beam may be usefully considered.

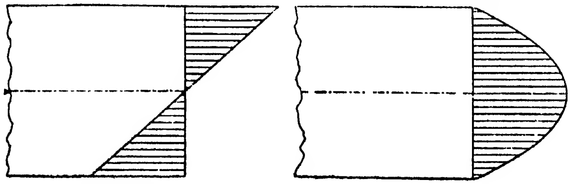


FIG. 12 FIG. 13

At any vertical section of such a beam under transverse loading, normal or tensile and compressive stresses, and tangential or shearing stresses are developed.

In conformity with the accepted theory of flexure, the intensity of normal stress on any vertical section of a beam varies with the distance from the neutral axis. Consequently the stress variation can be represented as in Fig. 12 by the ordinates to a straight line.

The intensity of tangential or shearing stress at any

vertical section of a beam is of maximum value at the neutral axis and diminishes to zero at the extreme fibres. In the case of a beam of rectangular section the shearing stress intensity varies as the ordinates to a parabola, as in Fig. 13, the maximum value at the neutral axis being $1\frac{1}{2}$ times the average value.

On any inclined section of a beam the intensities of normal and shearing stresses are not of the same values as those on vertical sections. Moreover, where shearing stress occurs on a vertical section, the maximum normal stress will be developed on an inclined section.

The formula given in text-books on applied mechanics shows that if t = horizontal tensile stress intensity, s = shearing stress intensity at any point in a beam, and $t_{(\max)}$ = maximum tensile stress intensity, the value of the latter will be

$$t_{(\max)} = \tfrac{1}{2}t + \sqrt{(\tfrac{1}{4}t^2 + s^2)}$$

The direction of this maximum tensile stress intensity will be at an angle equal to half the angle of which $\frac{1}{2}t/s$ is the cotangent, or $a = \frac{1}{2} \angle\ cot\ (\frac{1}{2}\ t/s)$.

It follows that at any point in a beam where shearing stress is of zero value, as at points of maximum bending moment and along the extreme fibres of the beam, the direction of the maximum tensile stress is horizontal; and that at any point where the horizontal tensile stress is of zero value, as at the neutral axis and at points of zero bending moment, the direction of the maximum tensile stress is at an angle of 45° to the horizontal, and the maximum tensile stress intensity is equal to that of the shearing stress at the same point.

Above the neutral axis of any section where the bending moment is of more than zero value, the direction of the maximum tensile stress is at an angle of more than 45° and changes until it is at an angle of 90° at the extreme upper or compression fibre.

Typical variations in normal, shearing, and maximum tensile stresses are illustrated diagrammatically in Fig. 14, where at any given vertical section AB, the variation of the fibre stress is shown by a diagonal straight line, drawn solid;

that in the shearing stress by a parabolic curve denoted by a broken line; and that of the maximum tensile stress by a line of dots and dashes.

The directions of the lines of maximum tension and compression in a rectangular beam, freely supported at the

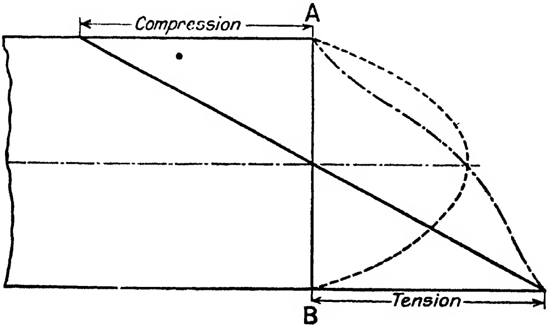


FIG. 14

ends, are illustrated in Fig. 15, which is based on one to be found in Rankine's *Civil Engineering*. It will be understood that the precise direction at any point is governed by the relations existing between shearing force and bending moment.

Functions of Tension Reinforcement.—The primary function of the steel used as reinforcement is to resist the principal tensile stress, the concrete being relied upon for resistance to compressive and shearing stresses.

Steel is also employed to aid the concrete in resisting compressive and shearing stresses and tensile stress on diagonal planes, but for the purpose of the present discussion it will be convenient to deal first with horizontal or longitudinal tension reinforcement, reserving other classes of reinforcement for subsequent consideration. Another important function of the reinforcement is to ensure the thorough distribution of strain in every part of the concrete in which it is embedded, thereby preventing the concentration of strain and consequent rupture at any point where there may be an element of weakness. If the steel were employed

in large units—such as bars of **T** or **I** section,—it would be impossible to attain the required result.

Disposition of Tension Reinforcement.—If theoretical principles were strictly followed, an ideal arrangement of the tension reinforcement would be based upon lines of maximum stress such as those denoted in Fig. 15 for a beam freely supported at the ends. It will be seen on reference to this diagram that at the middle of the beam, or point of maximum bending moment, and for some distance on either side of it the direction of the tensile stresses is horizontal, or nearly horizontal, close to the lower side of the beam. Therefore horizontal bars are clearly required. Towards the ends of the beam the diagonal tensile stresses acquire increased importance, and the concrete should be suitably reinforced

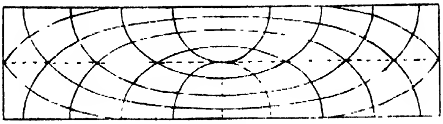


FIG. 15

for resistance to these stresses. Provision is frequently made, wholly or in part, for diagonal tension by bending up the ends of some of the horizontal bars, thereby complying in a measure with the arrangement suggested by the lines of stress represented in Fig. 15. This course is a good one for adoption, as there is comparatively little need for horizontal reinforcement at the ends of the beam, owing to the progressive diminution of bending moment as the distance from the middle of the beam is increased.

The effect of bent-up bars, however, is discussed in a succeeding chapter.

So far as our immediate purpose is concerned, we have decided that reinforcement consisting of longitudinal or horizontal bars, not too large in diameter, is suitable for effective resistance to horizontal tension, and that the ends of some of the bars may be bent up without detriment to the efficiency of the reinforcement for this purpose.

The disposition and amount of steel required must be

settled in each case in accordance with the distribution of stress, which can be ascertained by the usual analytical or graphical methods.

With regard to subdivision of the steel, it may be said that if absolute compliance with abstract theory were practicable, the proper course for adoption would be to employ steel wire of the smallest possible gauge, distributed so that numberless hair-like filaments of metal might be present in every part of the concrete requiring reinforcement against horizontal tension. But this method is not suited to the exigencies of everyday work. The best results are always obtained when designers make use of reinforcement in the form of reasonably small bars or rods, and give due consideration to all other essential requirements.

Theory of Flexure as Applied to Plain Concrete Beams.—The familiar theory of flexure stated in text-books is founded upon the hypotheses (1) that a plane cross section of a beam will remain plane during and after bending, and (2) that the structural material employed conforms with Hooke's Law as to the proportionality of stress to strain.

According to hypothesis (1), the deformation of the fibres at any section of a beam is proportional to the distance of the fibres from the neutral axis.

In beams subject to simple bending, resulting from the application of purely transverse loading, the neutral axis is situated at the centre of gravity of the beam section, while in members subject to combined stresses the neutral axis may be within the section or may be an imaginary line entirely outside the section, according to circumstances.

In accordance with hypothesis (2), the *stress* in the fibres at any section of a beam is proportional to the distance of the fibres from the neutral axis. Thus, as the *strain* in the fibres is also assumed to be proportional to their distance from the neutral axis, Hooke's Law is the basis of what is generally termed the *straight-line* theory of stress distribution in beams.

The straight-line theory is embodied in all practical formulæ for the flexure of beams, with the exception of some which

have been proposed for reinforced concrete beams, although wrought iron and steel are the only structural materials in general use following Hooke's Law with substantial accuracy up to the elastic limit.

It follows that any conclusions, based upon Hooke's Law, hypothesis (2), with regard to concrete can only be approximately correct. Moreover, the strict accuracy of hypothesis (1) and of any conclusions therefrom derived is still open to question.

Nevertheless, formulæ for plain concrete beams based upon the ordinary theory of flexure are sufficiently accurate for all practical purposes, and the only alternative to their use would be an exact analysis by the aid of stress-strain diagrams plotted from the results of tension and compression tests up to the limits of the actual stresses involved.

Low Efficiency of Plain Concrete Beams.— The low efficiency and lack of economy characterizing the plain concrete beam is due to the fact that the tensile strength of the material is about one-tenth of its compressive strength.

To make clear the effect of this difference of resistance, let us consider the case of a concrete beam of rectangular section supported at each end, the neutral axis passing through the centre of gravity of the section, as shown in Fig. 16. The

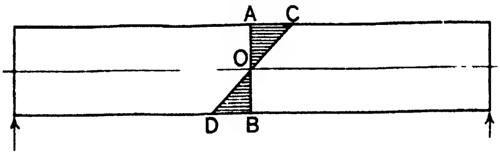


FIG. 16

stresses due to bending moment are equal above and below the neutral axis, and, together with their distribution in accordance with the hypotheses stated in the preceding paragraph, can be represented at any section of the beam by the triangular areas **AOC**, **BOD**. The maximum compressive stress is in the top fibres, and the maximum tensile stress in the bottom fibres of the beam, these stresses diminishing as

the neutral axis is approached, where the stress reaches zero value. Owing to the difference between the compressive and tensile strength of the concrete the lower fibres would fail in tension long before the upper fibres could fail in compression.

A similar though less noticeable inequality of compressive and tensile strength characterizes cast iron, and the familiar inverted **T**-section of a cast iron beam illustrates one method of making good a deficiency of resistance in the tension area.

As expressed by Rankine,* the object of the **T**-shaped section is "to economize any material whose resistance to cross-breaking by crushing and by tearing are different, by

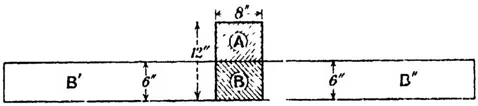


FIG. 17

so adjusting the position of the neutral axis that the tendencies of the beam to break across by crushing and by tearing shall be as nearly as possible equal."

Treatment of the same kind would be out of the question in the case of concrete, for if the difference of resistance were completely adjusted, the dimensions of the beam would assume absolutely impracticable proportions. This view is illustrated by Fig. 17, where (A) and (B) represent two halves of a concrete beam 8 in. wide by 12 in. deep. Assuming the neutral axis to be equidistant from the upper and lower surfaces, the compression area (A) would be 48 sq. in., and to provide a tension area of equal value, say, 480 sq. in., it would be necessary to increase the width of the lower portion (B) to an enormous extent. Thus, if two 6 in. flanges (B′) and (B″) were added, the total width of the beam would be 80 in. at the bottom.

The final dimensions of the beam would be quite impracticable, and otherwise objectionable. We therefore see

* *Civil Engineering*, 1898 Edition (p. 256).

that concrete alone does not lend itself to the economical construction of beams.

Efficiency Increased by Reinforcement.—By the incorporation of a relatively small percentage of steel in the tension area (**B**) of a rectangular beam such as that represented in Fig. 17, the lacking element of strength can easily be supplied without increasing the original sectional area of the beam. The compressive resistance of the concrete in the area (**A**) can then be fully utilized, and as the corresponding tensile resistance below the neutral axis is assumed to be supplied by the steel alone, the sectional area of the part (**B**) can be reduced if desired so as to effect further economy by converting the member into a **T**-shaped beam.

In Chapter I, under the head of General Characteristics, the inefficiency of plain concrete beams and the advantages to be obtained by the use of reinforcement are stated in general terms. The question of efficiency may now be considered a little more closely.

A comparatively small proportion of steel in tension is sufficient to permit the compressive strength of the concrete to be fully developed, but the proportion of steel required, and the extent to which the tensile resistance of the steel can be utilized in any given case, depend upon the elastic properties of the concrete and the steel, and upon the position of the neutral axis. Similarly, the position of the latter depends upon the elastic properties of the concrete and the steel, and either upon the stresses in these materials or upon the proportion of steel employed as reinforcement. Thus it will be seen that all the factors involved are in close relationship.

The diagrams given as Figs. 18 to 21 will suffice to make things clear without the employment of formulæ or calculations. The diagrams are based upon the elastic modular ratio, $m = 15$. In each diagram the line (**AB**) represents any vertical section extending from the upper surface of a beam to the centre of the reinforcement. The length of the line is equal to d, the effective depth of the beam, both of these dimensions being taken at unity.

A scale is given at the left-hand of each diagram, by the aid of which neutral axis depths n may be obtained in terms of d. The upper scale in every case denotes compressive stress in the extreme fibres of the concrete and the lower scale gives tensile stress in the steel as governed by the assumed value of m; the diagonal lines indicate compressive and tensile stresses in accordance with the straight-line theory of flexure.

In Fig. 18 the position of the neutral axis is constant at the centre of the line (AB) for all correlated stresses in the concrete and the steel, and the relationship between the compressive

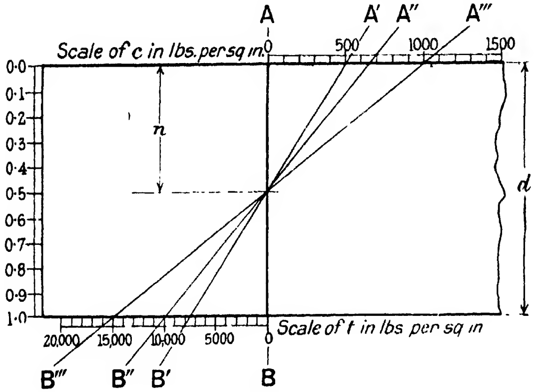


FIG. 18

and tensile stresses can be ascertained readily by drawing any lines, such as (A′B′), A″B″), and (A‴B‴).

For our present purpose, we will assume the permissible working stresses at $c = 500$ lb. per square inch for concrete in compression, and $t = 15{,}000$ lb. per square inch for steel in tension.

On this basis, line (A′B′) gives $c = 500$ lb. per square inch, and $t = 7{,}500$ lb. per square inch, the latter being just half the permissible stress for steel; line (A″B″) gives $c = 666$ lb. per square inch, and $t = 10{,}000$ lb. per square inch, the former beyond the stated limit and the latter only

two-thirds of the permissible stress; line (A''' B''') gives $c = 1{,}000$ lb. per square inch, double the working stress, and $t = 15{,}000$ lb. per square inch, which is exactly the permissible working stress.

Therefore, Fig. 18 demonstrates the point that if the position of the neutral axis is fixed arbitrarily it may be impossible to utilize both the concrete and the steel in an efficient and economical manner.

In Fig. 19 the tensile stress intensity in the steel is taken as constant, the compressive stress intensity in the concrete

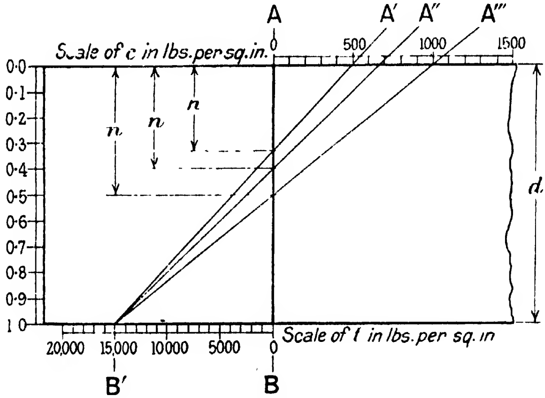


FIG. 19

and the position of the neutral axis being variable. In this case, for $t = 15{,}000$ lb. per square inch we have the following values—

Line (A' B')	$c =$ 500 lb. per sq. in.	and	$n =$	0·33
,, (A'' B')	$c =$ 666 ,, ,,	,,	$n =$	0·40
,, (A''' B')	$c =$ 1,000 ,, ,,	,,	$n =$	0·50

Here the line (A'B') denotes stresses in exact compliance with those permissible, the two values for c given by lines (A''B') and A'''B') being inadmissible.

In Fig. 20 the compressive stress intensity in the concrete is taken as constant, the tensile stress intensity in the steel

and the position of the neutral axis being variable. Then for $c = 500$ lb. per square inch, we obtain—

Line (A′ B′) $t = 5{,}000$ lb. per sq. in. and $n = 0·60$
„ (A′ B″) $t = 11{,}250$ „ „ „ $n = 0·40$
„ (A′ B‴) $t = 17{,}500$ „ „ „ $n = 0·30$

These results show that the efficiency of steel progressively increases with the rise of the neutral axis.

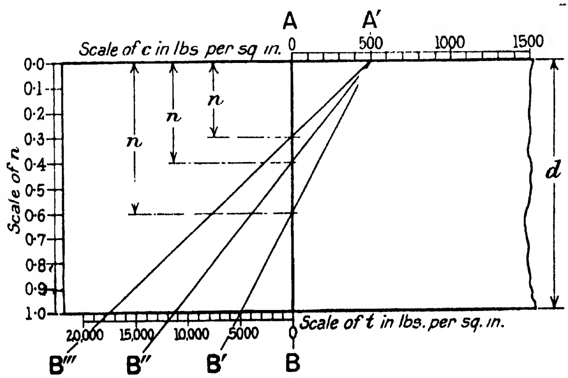


FIG. 20

Fig. 21 is a diagram showing that if the stress in the concrete is increased while the stress in the steel is decreased, the neutral axis will fall, and conversely that if the stress in the concrete is decreased, while the stress in the steel is increased, the neutral axis will rise.

The preceding diagrams are reproduced in a slightly altered form from a pamphlet describing an instrument* devised by the author for the calculation of reinforced concrete beams of rectangular and T-sections, each setting of the instrument resulting in an arrangement of the movable parts which constitutes a stress diagram denoting the stress intensities and correlative values needed by the designer.

Having now before us a clear idea of some of the variable

* Manufactured by Messrs. J. Halden & Co., Ltd., London and Manchester.

factors involved in the application of reinforcement to concrete beams, we will revert to the case of the plain concrete beam previously considered, the neutral axis being at the

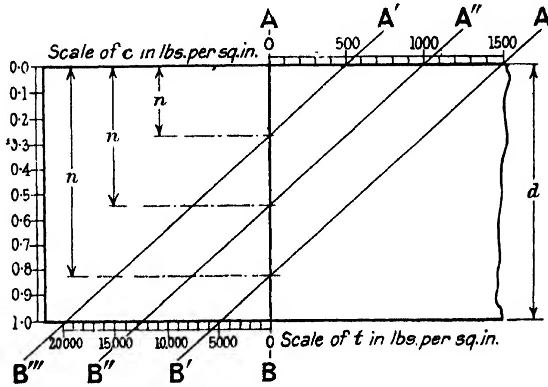


FIG. 21

centre of the section. This beam measuring 8 in. by 12 in., has a sectional area of 96 sq. in., and is here represented by Fig. 22. The resistance moment, calculated by the ordinary

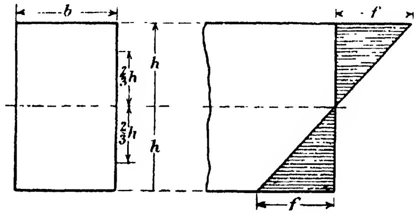


FIG. 22

formula $R = fbh\frac{2}{3}h$, is 9,600 inch-pounds, if the extreme fibre stress in the concrete is limited to 50 lb. per square inch.

Thus, taking $f = 50$ lb. per square inch, $b = 8$ in., and $h = 6$ in., we have

$$R = (50 \times 8 \times 6 \times 4) = 9{,}600 \text{ inch-pounds.}$$

We will now consider three different methods of employing

an equivalent sectional area of concrete, 96 sq. in., in the form of a reinforced concrete beam.

Method I.—Still assuming the neutral axis to be at the centre line of the beam section, we will place the reinforcement so that the centre of tension in the bars coincides with the bottom of the concrete, as in Fig. 23, so as to avoid altering the originally stated depth of the beam. Then the

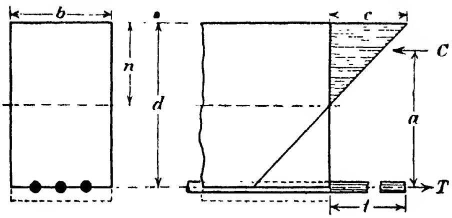


FIG. 23

total compression in the concrete, taking $c = 500$ lb. per square inch as before, and the mean stress at 250 lb. per square inch in the area $b{\cdot}n = 8$ in. by 6 in., will be $C = (250 \times 8 \times 6) = 12{,}000$ pounds. This force may be considered as acting at the centroid of the triangular stress area, which in the present case is 2 in. below the top of the beam. Therefore, as the resistance of the concrete in tension is neglected, the arm of leverage (a) of the internal forces has a length of 10 in. Multiplying the compressive force by the length of the lever arm, we obtain the resistance moment of the beam in compression, or $R = (12{,}000 \times 10) = 120{,}000$ inch-pounds.

We have now to provide sufficient steel to give equal resistance in tension. From Fig. 18 it has been found that with the neutral axis in the stated position the tensile strength of the steel cannot be utilized beyond a stress of $t = 7{,}500$ lb. per square inch, without exceeding the limit $c = 500$ lb. Consequently, the required area of steel will be $A = (12{,}000 \div 7{,}500) = 1{\cdot}6$ sq. in. The total tension in the steel will be $T = (7{,}500 \times 1{\cdot}6) = 12{,}000$ lb. And for the resistance moment in tension we obtain: $(7{,}500 \times 1{\cdot}6 \times 10) = 120{,}000$ inch-pounds, as before.

The comparison shows that the reinforced concrete beam possesses more than twelve times the strength of the corresponding plain concrete beam. It must be pointed out, however, that the total depth of the reinforced beam must be made, say, 13 in., so as to provide cover for the steel bars as shown by dotted lines in Fig. 23. The concrete so employed might be taken from the sides of the section below the neutral axis and thus leave the sectional area equal to that of the plain beam, but even if the required area of 8 sq. in. were to constitute an addition to the section, its cost would be only a small item.

Method II.—This time we will base calculations upon the full development of the permissible stresses, $c = 500$ lb. per square inch for concrete, and $t = 15{,}000$ lb. per square inch for steel. Then, as given by Fig. 19, the neutral axis will be at the distance $0{\cdot}3d$ from the top of the beam, as in Fig. 24.

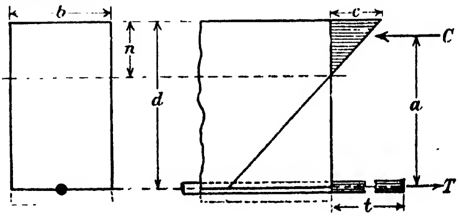


FIG. 24

Proceeding as before, we find the resistance moment of the beam in compression to be $R = 85{,}333$ inch-pounds, and that to provide for equal resistance in tension the area of the reinforcing steel must be 0·53 sq. in.

Although the strength of the reinforced beam is barely nine times that of the plain beam (instead of over twelve times, as in Method I), this result is attained by the use of one-third the quantity of steel. Hence the beam as here designed is really more efficient and more economical than that provided by Method I.

The note previously made as to the concrete required as cover for the steel bars also applies to this case.

Method III.—We will now assume the concrete sectional

area of 96 sq. in. to be distributed in the form of a **T**-beam, having a rib 4 in. wide by 9 in. deep to the centre of the reinforcement, and a compression flange 18 in. wide by 3 in. thick. After providing cover for the steel, we still have 2 sq. in. of concrete to spare, and this may be allocated to fill in the angles at the junction of the rib and flange as shown in Fig. 25. Taking the permissible working stresses

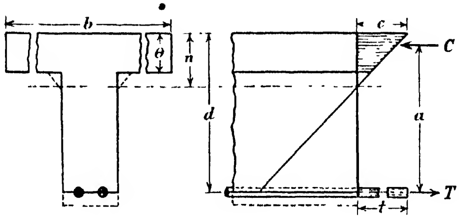


FIG. 25

for concrete and steel at 500 lb. and 15,000 lb. per square inch as before, the neutral axis will again be at the distance $0{\cdot}3d$ from the top of the beam, and therefore 1 in. below the under side of the flange.

Consequently, as represented in Fig. 25, the compressive stress diagram is trapezoidal instead of triangular in form, and the mean stress in the concrete is proportionately increased.

Calculations made in accordance with the formulæ given in a later chapter show that the resistance moment of the beam in compression is 180,000 inch-pounds, approximately, and that to provide equal resistance in tension, 1·13 sq. in. of steel will be necessary.

The strength of the beam as designed is nearly twenty times that of the original plain concrete beam, a result attained by only about two-thirds the proportion of steel employed under Method I.

The foregoing comparisons have been made mainly with the object of illustrating the high efficiency and economy of reinforced as compared with plain concrete, but also for the purpose of showing that there is ample scope for ingenuity in the design of reinforced concrete beams.

Theory of Flexure for Homogeneous Beams.—As

an introduction to the subject of this section, it will be helpful, especially to those who have not made a special study of reinforced concrete, to consider briefly the general equation for the resistance moment of rectangular beams, as derived from the ordinary theory of flexure for homogeneous materials, with the objects of finding out how far the equation is applicable to reinforced concrete, and of showing the modifications which are necessary for the purpose of making it so applicable.

As the ordinary theory of flexure is founded upon the straight-line law, the same basis is adopted in connection with the reinforced concrete beam formulæ derived in the course of the discussion.

Ordinary Formula for Resistance Moment.—As very generally stated in text-books, the resistance moment of solid rectangular beams is—

$$R = \frac{f \cdot I}{h} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and as the inertia moment $I = \frac{b \cdot d^3}{12}$, and $h = \frac{d}{2}$, the equation is often written—

$$R = \frac{f \cdot b \cdot d^2}{6} \quad . \quad . \quad . \quad . \quad . \quad (b)$$

An equivalent form of the same equation is—

$$R = f \cdot b \cdot h \cdot \tfrac{2}{3} h \quad . \quad . \quad . \quad . \quad (c)$$

The notation in formulæ (a), (b), and (c) is explained, with the exception of the symbol I, in Fig. 26, which includes a cross section and a part longitudinal section, with stress diagram, of a rectangular beam of any homogeneous material. In this illustration f denotes the extreme fibre stresses, and the two halves of d are denoted by the symbols h' and h, respectively.

It will be noticed that the formulæ do not suggest any mental picture of their purport and operation, and that they do not lend themselves to the preparation of explanatory diagrams. The meaning of (c) is certainly a little more obvious than that of (a) and (b), but like the other two, it

does not differentiate between the extreme fibre stresses (f) in compression and in tension, and it evidently deals with only one-half (which may be either $b \cdot h'$ or $b \cdot h$) of the beam section.

Of course, the three forms of the general equation are perfectly correct and give identical results. The foregoing

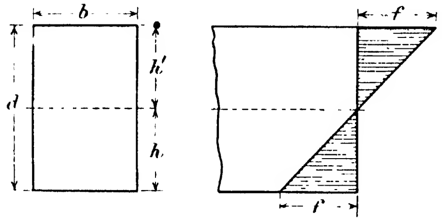


FIG. 26

remarks upon the three methods of expression are simply made for the purpose of showing that none of the formulæ has the appearance of being adaptable for use in the design of reinforced concrete beams.

Modification of Ordinary Formula for Resistance Moment.—We will now see how formula (c) can be modified so as to

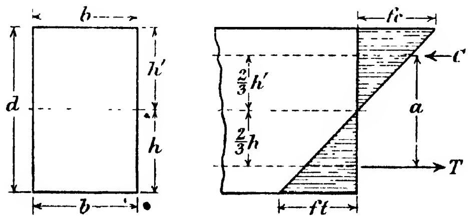


FIG. 27

make it more expressive. In the first place, as diagrammatically illustrated in Fig. 27, the fundamental principles of flexure are that the total compressive force (C) above the neutral axis is equal to the total tensile force (T) below the neutral axis, the two forces forming a *couple*, the moment

of which, termed the *resistance moment*, is obtained by multiplying either force into the *lever arm* (a) of the forces.

The total compressive force (C) acting upon the area ($b \cdot h'$) of the cross section, is the product of the *mean* compressive stress and the area ($b \cdot h'$). The compressive stress fc attains maximum value at the extreme fibres, and is of zero value at the neutral axis. Consequently for a triangular stress area, $\frac{1}{2}fc$ denotes the mean compressive stress, and $C = (\frac{1}{2}fc \cdot b \cdot h')$.

Similarly for the total tensile force we have $T = (\frac{1}{2}ft \cdot b \cdot h)$

Each of the forces (C) and (T) may be regarded as acting at the centroid of the corresponding triangular stress area, the distances of the two points from the neutral axis being $\frac{2}{3}h'$ and $\frac{2}{3}h$, respectively. The length of the lever arm of the internal forces is equal to $(\frac{2}{3}h' + \frac{2}{3}h)$, and as $d = (h' + h)$, we obtain: $a = \left(d - \frac{h' + h}{3}\right)$.

Referring to Fig. 27, it will be readily understood that the resistance moment of the beam in compression can be ascertained by using the force (C) and multiplying this quantity by the lever arm a, thereby taking the moment about the centre of tension.

Whence, in its most simple form, the equation is—

$$R = C \cdot a \quad . \quad . \quad . \quad . \quad . \quad .$$

Substituting the symbols represented by C and a, as stated above, the equation becomes

$$R = (\tfrac{1}{2}fc \cdot b \cdot h')\left(d - \frac{h' + h}{3}\right) \quad . \quad . \quad . \quad \text{(d)}$$

Proceeding on similar lines, for the resisting moment in tension we have—

$$R = T \cdot a \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

and

$$R = (\tfrac{1}{2}ft \cdot b \cdot h)\left(d - \frac{h' + h}{3}\right) \quad . \quad . \quad . \quad \text{(e)}$$

It should be noted that the two internal forces (C) and (T), parallel to each other and acting in opposite directions, must be equal in order to secure equilibrium in the beam. Further,

the beginner should guard against any misinterpretation of the axiom that the resistance moment of a beam is in all cases equal to the bending moment. The axiom is literally true, and the equality of the two moments exists from the stage where the applied loading is of zero value to the stage where the loading is of such magnitude as to result in the rupture of the structural material, either by tension or by compression. At this final stage both the moments obviously disappear. In practical work, however, the designer starts with a specified or calculated bending moment for a given beam, which he must design so that the calculated *safe* resistance moment shall be at least equal to the predetermined bending moment. Thus, to ensure safety, the *actual* or *ultimate* resistance moment must be considerably greater than the specified bending moment for any beam.

To illustrate the practical effect of the formulæ quoted and derived above, we give below calculations for a rectangular beam of plain concrete, where $b = 8$ in., h and $h' = 6$ in., $d = 12$ in. Assuming that the maximum fibre stress must not exceed 50 lb. per square inch, we have f, fc, and $ft = 50$ lb. per square inch. Then

By (a)

$$R = \frac{50\left(\frac{1}{12} \times 8 \times 1{,}728\right)}{6} = 9{,}600 \text{ inch-pounds.}$$

By (b)

$$R = \frac{50 \times 8 \times 144}{6} = 9{,}600 \text{ inch-pounds.}$$

By (c)

$$R = 50 \times 8 \times 6 \times 4 = 9{,}600 \text{ inch-pounds}$$

By (d) and (e) for compression and tension, respectively,

$$R = (\tfrac{1}{2}\, 50 \times 8 \times 6) \times 8$$
$$= 1{,}200 \times 8 = 9{,}600 \text{ inch-pounds.}$$

Derivation of Formulæ for Resistance Moment of Reinforced Concrete Beams.—Having in (d) and (e) two formulæ expressing fundamental principles in a suitable manner, our next task is to adapt them for reinforced concrete beam calculations.

Let Fig. 28 represent a reinforced concrete beam of the same dimensions as those employed in the foregoing calculations, namely: $b = 8$ in., and $d = 12$ in. The reinforcement consists of a steel bar A placed so that the centre of its cross section is intersected by the bottom line of the beam. This arrangement is assumed so that the *effective depth* of the beam, (d), is the same as the *total depth* of the homogeneous beam before considered. It will be

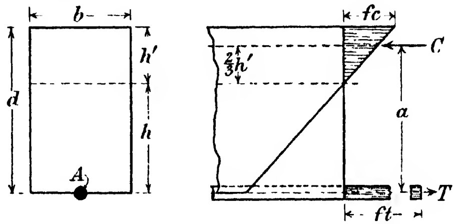


FIG. 28

understood that a practical beam would require the addition of an extra inch or so in depth, so as to provide cover for the steel. Adopting the safe working stresses $fc = 500$ lb. per square inch and $ft. = 15{,}000$ lb. per square inch, the position of the neutral axis will not be at the centre line of the section but such that $h' = \frac{1}{2}h$ (see Fig. 19).

Consequently the point of application of the tensile force (T) instead of being at the distance $\frac{1}{3}h$ from the bottom, as in Fig. 27, is at the bottom of the beam; the point of application of the compressive force (C) is higher than in Fig. 27 owing to the reduced height of (h'); and for the length of the lever arm we have $a = (h + \frac{2}{3}h') = \left(d - \frac{h'}{3}\right)$.

Formula (d) now becomes

$$R = (\tfrac{1}{2}fc{\cdot}b{\cdot}h')\left(d - \frac{h'}{3}\right) \quad . \quad . \quad . \quad . \quad (f)$$

As the total tensile force (T) is now applied at the centre of the steel bar, and as the bar is of small sectional area in comparison with the tensile force, the value of (T) may be represented without appreciable error by the product

of the tensile stress intensity in the steel and the sectional area (A) of the bar, or $T = ft.A$. Subject to this modification, formula (e) becomes

$$R = (ft \cdot A)\left(d - \frac{h'}{3}\right) \quad . \quad . \quad . \quad . \quad (g)$$

Using values as above stated, namely: $fc = 500$ lb. per square inch, $b = 8$ in., $d = 12$ in., $h = 8$ in., and $h' = 4$ in., we have—

By (f)

$$R = (\tfrac{1}{2}\ 500 \times 8 \times 4)\left(12 - \frac{4}{3}\right) = 85{,}333 \text{ inch-pounds.}$$

The area of steel required for the development of the tensile force on the basis $ft = 15{,}000$, is $A = 0{\cdot}53$ square inches.

Therefore by (g) we get

$$R = (15{,}000 \times 0{\cdot}53)\left(12 - \frac{4}{3}\right) = 85{,}333 \text{ inch-pounds.}$$

Having now derived formulæ for reinforced concrete beams from the ordinary resistance moment equation, we will next transpose these formulæ into standard notation and indicate alternative methods of expression.

The only changes necessary in (f) are the substitution of c for fc and the replacement of h' by n = distance of the neutral axis from the extreme fibres in compression. Formula (f) thereby becomes

$$R = \tfrac{1}{2}\, c \cdot b \cdot n\left(d - \frac{n}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad (fa)$$

In the practical design of beams, it is very convenient to employ tables or diagrams giving values for variable factors, or one value for a group of such factors, under stated conditions.

Thus, if the factor n in (fa) is expressed as a ratio of d, we have $n_1 = n/d$, and the formula may be written

$$R = b \cdot d^2\left[\tfrac{1}{2}\, c \cdot n_1\left(1 - \frac{n_1}{3}\right)\right] \quad . \quad . \quad . \quad . \quad (fb)$$

Again, for employment with tables and diagrams whence values for the group $\left[\tfrac{1}{2} c \cdot n_1\left(1 - \frac{n_1}{3}\right)\right]$ can be obtained for

any given set of conditions, we can denote the value of the group by the symbol Q and reduce (fb) to

$$R = Q \cdot b \cdot d^2. \qquad \text{(fc)}$$

Similarly, in (g), the substitution of t for ft and n for h' gives

$$R = t \cdot A\left(d - \frac{n}{3}\right) \qquad (ga)$$

Further, expressing A as a ratio in terms of $b \cdot d$, we have $r = A/b \cdot d$, and substituting $\left(1 - \frac{n_1}{3}\right)$ for $\left(d - \frac{n}{3}\right)$ we obtain

$$R = b \cdot d^2\left[t \cdot r\left(1 - \frac{n_1}{3}\right)\right]. \qquad (gb)$$

As in the case of (fc) we can also write

$$R = Q \cdot b \cdot d^2 \qquad (gc)$$

Dissimilar Expression of Equivalent Resistance Moment Formulæ.—Everyone who has had occasion to read different books on the subject of reinforced concrete must have experienced much inconvenience in consequence of the widely differing systems of notation and methods of expression employed by the authors responsible.

The idea of establishing standard notation for engineering formulæ in order to obviate the confusion and waste of time caused by the haphazard employment of symbols was brought forward more than 30 years ago by Professor Jamieson, and has been discussed since then by the Institution of Electrical Engineers, the Civil and Mechanical Engineers' Society, the Engineering Standards Committee, and the International Electro-Technical Commission. At a meeting of the Civil and Mechanical Engineers' Society in 1908, the author opened a discussion on the subject, and the matter was subsequently taken up by a Special Committee, but the project unfortunately lapsed owing to the pressure of other work connected with the joint incorporation of the Society and the Society of Engineers. Nothing appears to have been done by the other bodies mentioned, and the credit for formulating codes of standard notation for reinforced concrete, structural engineering and other branches

of pure and applied science is due to the Concrete Institute. In 1909, the main principles put forward by Mr. E. Fiander Etchells, A.M.Inst.C.E., were adopted by the Institute, and the Report of a Committee was published, together with schedules of standard symbols.* The standard notation has been adopted by the Royal Institute of British Architects, the Institution of Civil Engineers, and in various regulations embodied in Acts of Parliament; and is beginning to find its way into reinforced concrete and other engineering literature. As stated elsewhere in this work, the standard notation of the Concrete Institute has been modified to some extent since the publication of the Report mentioned, and the Author has adopted the notation in its most recent form.

Some typical examples of the divergencies which cause so much loss of time to professional men and so much perplexity in the minds of students are given in Table IV, where the equivalence of the formulæ quoted is proved by their transcription in standard notation and their expression in standard form.

Classification of Flexure Theories.— The many theories of flexure which have been evolved from time to time for reinforced concrete beams, and the formulæ resulting from the same theories, constitute a most interesting subject for study, and it would be quite easy to fill a good-sized volume with an account of the experimental investigations, the data deduced therefrom, and the mathematical reasoning forming the basis of the theories and resultant formulæ.

A complete record of the kind, however instructive it might be to the student, would be of no great value to the practical designer because several of the theories in question have been either abandoned by their authors or have failed to secure general approval. Others are unsuitable except for laboratory investigations, and of the remainder only one—the *straight-line* theory—has been generally adopted by engineers and public authorities throughout the world.

* *Mnemonic Notation for Engineering Formulæ.* (London: E. & F. N. Spon, Ltd., 1918.)

For these reasons, a brief review of the subject will serve the purpose of this work.

Ordinary Beam Theory with Arbitrary Modifications.—In the early days of reinforced concrete, long before the true principles of the combination were understood and when the need for taking into account the elastic modular ratio for the constituent materials was not generally recognized, the methods of calculation adopted by pioneers were based in the main upon the ordinary theory of flexure for homogeneous materials with modifications suggested by practical experience.

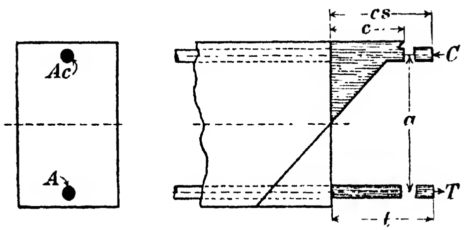


FIG. 29

Many formulæ of the kind embody theoretical errors of more or less serious character. In spite of this theoretical drawback, some of them have been employed for a long period of years in the design of thousands of structures with entirely satisfactory results, and are still in extensive use—a result which is probably attributable to the wide experience and intuition of the engineers making use of such formulæ.

A theory adopted by several Continental firms is illustrated in Fig. 29. By the employment of steel bars of equal sectional area in tension and compression, it is assumed that the neutral axis occupies a central position, and that the required sectional areas of steel may be computed for equal stresses in the same way as the flanges of a steel girder. As the compressive as well as the tensile strength of the concrete is neglected, the effective resistance of the beam is assumed to be due entirely to the steel, the duty of the concrete being merely to provide a connecting web between the two areas of steel.

In accordance with this theory, the equation for the resistance moment is: $R = cs \cdot A \cdot a = t \cdot A \cdot a$.

Beams so designed cannot be regarded as constituting genuine reinforced concrete, and this method of trying to utilize the ordinary flexure theory in reinforced concrete beam design, embodies two theoretical errors: (1) Unless the steel bars were loosely inserted in holes formed in the concrete—a procedure quite inadmissible for the reason that the concrete would thereby be unable to act as a connecting web—the neutral axis could not be at the centre line of the section. If the area of the compression reinforcement is equal to that of the steel in tension, the neutral axis must be above the centre line. (2) The compressive resistance of the concrete cannot be got rid of by an act of volition on the part of the designer, and must always take a considerable proportion of the stress above the neutral axis. Therefore, the compressive resistance of the steel cannot be fully developed.

In practical work there is no objection to this method on the score of safety, providing the ultimate compressive resistance of the concrete is not exceeded. Moreover, in beams where compression reinforcement is required, the steel can be applied with economy under this method, for the reason that the permissible stress is not limited to m times the stress in the concrete.

Some methods of calculation based upon modifications of the ordinary flexure theory embody the error of equating the *moments* of the internal forces instead of equating the *forces* themselves. Of course, if the forces and the distance from the neutral axis of the point at which each force may be regarded as acting happen to be equal, the result will be the same as that given by the correct procedure. But any inequality of the forces, and any deviation of the neutral axis from the centre line of the beam section will inevitably lead to erroneous results. Formulæ coming under this category usually omit to take account of the elastic properties of the concrete and the steel respectively, and involve the misconceptions that each stress area can be treated independently, and that the full resistance of the steel in

compression and in tension can be developed simultaneously, the latter being a condition obtaining only when the neutral axis is abnormally low and when the compressive stress in the concrete is far beyond safe limits.

One formula of the kind even involves the assumption that the compressive stress area is of rectangular form, the stress intensity at the neutral axis being taken as equal to that at the extreme fibres. This curious hypothesis, however, is corrected in practice by taking the permissible stress in the concrete at about half the value of the extreme fibre stress customarily allowed.

While admitting the fact that empirical formulæ of different kinds have been employed with undoubted success for a quarter of a century or more by practical experts, knowing almost exactly what modifications are required to meet special cases, equations of this class cannot be recommended for general use by engineers and architects.

Theories Based upon Variable Values of Ec.—Flexure theories coming under this head are by no means in agreement as to the distribution of stress in the tension area, but with one exception they agree in representing variations of compression by a parabolic curve. Figs 30 and 31 illustrate two theories based upon the variation of the elastic modulus Ec in compression with the increase of stress, and upon the hypothesis that the modulus is of the same value in tension as in compression within limits denoted in the diagrams by a change in the direction of the tension stress-strain curve. In one diagram the direction changes gradually, and in the other it alters suddenly, but in each case the parallel portion of the tension area corresponds with the condition $Ec = 0$, the assumption being that the concrete is capable of extension without actual rupture and without an increase of stress. Theories involving this view of the extensibility of concrete have been shown by subsequent investigations to be untenable.

A theory somewhat akin to that illustrated by Fig. 31 is diagrammatically represented in Fig. 32, the value of Ec both in compression and in tension being taken as

constant as far as the points where changes of direction take place in the curves. Beyond the point in the compression area, the value of *Ec* is still constant at its reduced value and beyond the point in the tension area the value is $Ec = O$.

Another theory, illustrated in Fig. 33, is based upon the variation of the elastic modulus with stress intensity, and upon the value of the modulus being lower in tension

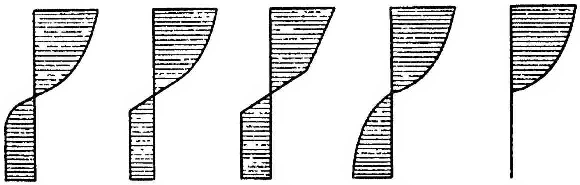


Fig. 30 Fig. 31. Fig. 32 Fig. 33 Fig. 34

than in compression, both of these conditions being in accordance with experimental data, although the precise relation between the value of the modulus in tension and in compression remains to be determined.

Fig. 34 represents what is commonly described as the *parabolic* theory, which is based upon the variation of the elastic modulus in compression and disregard of any tensile resistance by the concrete.

Theories Based upon Constant Values of Ec.—Flexure theories coming under this head are illustrated in Figs. 35 to 39, which are fairly self-explanatory. The erroneous hypothesis embodied in Fig. 35 that the elastic modulus possesses the same constant value, both in compression and in tension, dates from the early days of reinforced concrete. Three different views of the lower value of the elastic modulus in tension are represented in Figs. 36, 37, and 38. In the first of these the value is constant throughout; in the second the value corresponds with that in compression until the direction of the curve is suddenly changed, and beyond that point the reduced value is still constant; and

in the third the value of the modulus in tension is equal to that in tension as far as the point beyond which $Fc = O$.

Fig. 39 illustrates the generally adopted form of the

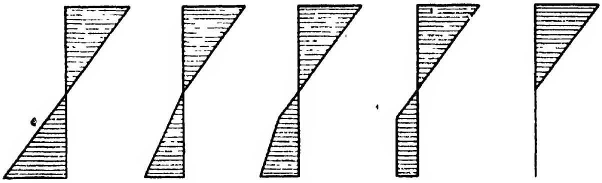


Fig. 35. Fig. 36 Fig. 37. Fig. 38. Fig. 39.

straight-line theory, where the value of Ec in compression is constant and the tensile resistance of the concrete is entirely neglected.

Parabolic and Straight-Line Formulæ.—In laboratory investigations into the behaviour of reinforced concrete beams under test loads it is often necessary to determine the tensile stresses up to the stage of loading when the first crack is observed in the concrete, and for such purposes, as well as for tests to destruction, formulæ are required wherein account is taken of the tensile resistance of the concrete. Equations of the kind may be based upon the straight-line or the parabolic distribution of stress, the latter being more accurate because it corresponds more closely to experimental data, especially when the limits of ordinary working stress have been passed.

In the design of reinforced concrete beams, formulæ including consideration of the tensile resistance of concrete are not necessary and should never be employed. Consequently from among the numerous sets of rules which have been derived from theories such as those briefly discussed in the preceding paragraph only two varieties remain for adoption in ordinary practice, namely: formulæ based upon the parabolic theory, and formulæ based upon the straight line theory.

The effect of the parabolic theory of stress distribution where applied to the design of beams may be gathered

from the series of diagrams given as Figs. 40 to 43. These diagrams are based upon the results of the exceedingly valuable investigations of Professor Talbot at the University of Illinois, and also serve the purpose of enabling the reader to compare the parabolic and straight-line theories at four different stages of loading, at which the deformation of the extreme fibres of the concrete ranges from one-fourth of its limit in compression up to the ultimate limit.

The parabolic curves NB represent the distributions of

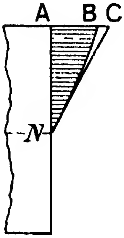


Fig. 40.

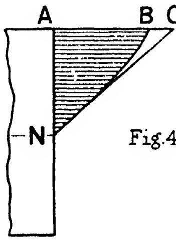


Fig. 41

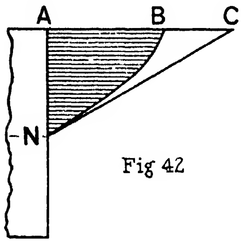


Fig 42

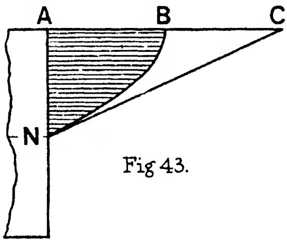


Fig 43.

stress corresponding with variations in the value of the elastic modulus, and the straight lines, NC, represent distributions of stress in accordance with an assumed constant value of the elastic modulus. In each diagram N denotes the position of the neutral axis, AB and AC representing the extreme fibre stresses according to the two theories, respectively.

However useful it may be for laboratory operations involving tests where the stresses developed are higher than those permitted in practical work, the parabolic theory

offers no appreciable advantage to the designer. This point is made clear by Fig. 40, where, for a deformation equal to one-fourth the limit in compression, the maximum fibre stress is about 0·45 per cent. of the ultimate resistance of the concrete, and there is very little difference between the lines **NB** and **NC**. Still less difference exists between the same lines when the deformation is such that the maximum fibre stress is 0·25 per cent. of the ultimate resistance of the concrete.

Consequently, within the limits of permissible working stresses there is very little to be gained by adopting the parabolic theory, which requires almost exactly the same proportion of steel as tension reinforcement, and offers only a small advantage in respect of the concrete.

Formulæ on the straight-line theory are less complicated than those derived from the parabolic theory, and any errors due to straight-line formulæ simply add to the factor of safety.

Compression Reinforcement in Beams.— Although steel is always more economically applied in the form of tension reinforcement, its employment as compression reinforcement in beams and kindred members is often desirable.

Steel can be so used with advantage in cases where relatively heavy loads have to be carried by beams whose external dimensions must be kept within limits governed by structural and architectural requirements.

Moreover, reinforcement is necessary on both sides of the neutral axis of any beam subject to alternations of stress, and in such a case the designer naturally takes full advantage of the steel by including in his calculations the additional resistance offered by compression reinforcement. Fig. 44 represents diagrammatically the case of three continuous beam spans where reversals of stress may occur in consequence of unequal loading, and the duties of the tension and the compression reinforcement are liable to temporary interchange. Therefore it will be seen that the adoption of compression reinforcement may be doubly justified.

It should be borne in mind that the stress in steel used as compression reinforcement must be limited to m times

·n ·the concrete, in order that the
square inch le to work together in the required
the permissi
stress in the ng the application of steel in com-
tion, values what has already been said concerning
inforced concrete.
udents, however, it may be useful
r a little more fully.
ns under which compression rein-
l upon to withstand tension owing to
vill first inquire whether the employ-
ssion can be avoided by an increased

Example
Tension.—1
or A = 1,
permits the
in Fig. 45.

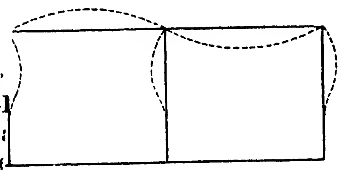


FIG. 11

. tension, in cases where it is required
l resistance without a corresponding
nsions of a beam.
t this can certainly be done, but not
y. Taking it as an essential condition
e stress in the concrete shall be kept
any increase in the proportion of tension
volve reduced economy owing to the fact
lowering of the neutral axis will make it
that the consequent safe tensile resistance of the
impossible to develop ful permissible resistance of the
steel. On the other hand, utilization of an increased
steel were taken as a basis would be accompanied by
amount of tension reinf the concrete.
excessive compressive substitution of a T-section for
It may be noted uld permit a larger amount of
a rectangular se to be employed advantageously,
tension reinfor

owing to the larger area of co
compressive stress. lesigner. This point

But we are now acting upon a deformation equal
increase is permissible in the width the maximum fibre
form of a compression flange. Mctimate resistance of
employing compression reinforcemeference between the
sidered in connection with **T**-beams, exists between the
with a beam of **T**-section, any prop that the maximum
its load-bearing capacity would involtimate resistance of
nature with those presented by the cas

With the object of explaining tpermissible working
additional steel as tension reinforcerd by adopting the
pression reinforcement alone, or as t exactly the same
reinforcement in a beam design alrecent, and offers only
performance of a given duty in the mrete.
we give below a few numerical exare less complicated
the aid of formulæ which will be foutory, and any errors

In comparing these results, it musthe factor of safety.
they are merely intended to illustrals.— Although steel
and not to establish fixed relationshipshe form of tension
methods of applying additional steel, ession reinforcement
of a constant ratio being out of the desirable.
variability of the principal factors invl cases where rela-

In the subjoined examples and di by beams whose
tension reinforcement, Ac = area of cn limits governed
ment, b = breadth of beam, d = effecents.
n = depth of neutral axis from top of both sides of the
depth of centre of compression reinforermations of stress,
beam, and here taken at $0{\cdot}1d$. The arkes full advantage
are not stated in square inches, but are ns the additional
expressing their relationship to the denoted by numbers
steel in Example I, which is taken proportionate amount of
specific measurements are not given at $A = 1$. Similarly,
relative proportions of these being for b, d, n, and i, the
The compressive stresses in the conc shown in the diagrams.
the tensile stress in the steel are stated te and the steel, and
pounds per square inch. The permissible n numbers denoting
to 600 lb. per square inch for concrete and stresses are limited
16,000 lb. per

square inch for steel. In the case of steel in compression the permissible stress is limited to m times the permissible stress in the concrete. In order to avoid needless complication, values are given in round numbers, as far as possible.

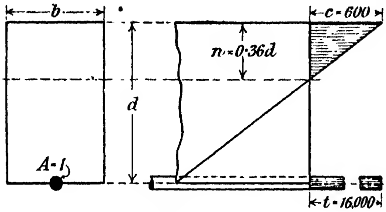


FIG. 45

Example I. Beam with "Economic" Proportion of Steel in Tension.—Here the area of steel in tension is taken at unity, or $A = 1$, and represents the "economic" proportion which permits the working stresses to be fully developed as shown in Fig. 45. The relative value of the resistance moment as

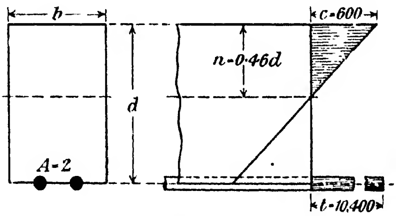


FIG. 46

calculated is expressed as $R = 100$ for the purposes of comparison.

Example II. Beam with Excess Proportion of Steel in Tension.—In this case, as represented in Fig. 46, double the proportion of steel is used in tension, with the result that the neutral axis takes a lower position. Consequently the larger compressive stress area increases the total compression, while on the tension side, the usefulness of the larger area of steel is impaired by the fact that the working

stress must be limited to 10,800 lb. per square inch, in order to avoid overstressing the concrete in compression. For these reasons the resistance moment is not greatly increased, its relative value being $R = 124$.

Example III. Beam with "Economic" Proportion of Steel in Tension, and an Equal Proportion Added in Compression.—

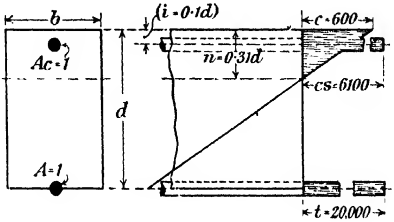


FIG. 47

Here, in addition to $A = 1$ of steel in tension, we have $Ac = 1$ of steel in compression. Proceeding first upon the assumption that the safe resistance of the concrete may be fully developed, we find this to be inadmissible, because, as shown in Fig. 47, the corresponding tensile stress in the steel, owing to the higher position of the neutral axis,

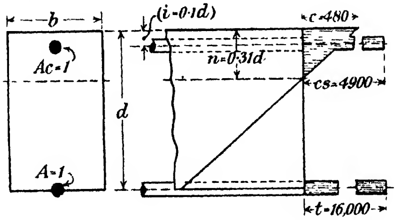


FIG. 48

would be over 20,000 lb. per square inch. Even if this were permissible, the resistance moment would be only $R = 126$, approximately.

Taking the limiting stress of 16,000 lb. per square inch for the steel in tension as a basis, we next find as represented in Fig. 48 that the maximum compressive stresses in the concrete and steel above the neutral axis are reduced

480 lb. and 4,900 lb. per square inch respectively, and at for the resisting moment we must be content with = 102. Therefore no appreciable increase of resistance gained by the extra area of steel employed.

Example IV. Beam as in Example III, but with Increased roportion of Steel in Tension.—As the beam in accordance th Example I permitted the full working stresses in both

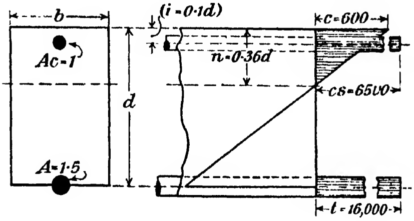


FIG. 49

the concrete and the steel to be attained, it is obviously desirable to avoid disturbance of these conditions when providing for the addition of compression reinforcement. Referring to Fig. 49, let us start, as in Fig. 45, with $A = 1$, and then add $Ac = 1$ as in Fig. 47. Next, to counterbalance Ac and to avoid any disturbance of the neutral axis, we will increase the amount of steel in tension to $A = 1{\cdot}5$, the whole of this acting at a stress of nearly 16,000 lb. per square inch, and $Ac = 1$ acting at 6,500 lb. per square inch. The result is that for the resistance moment we have the relative value $R = 142$, approximately.

Comparison of Examples I to IV.—To enable the results to be seen at a glance, they are stated below in tabular form—

Example No.	Relative Areas of Steel (total).	Relative Values of R.	Relative Values of R per Unit Area of Steel.
1	1	100	100
2	2	124	62
3	2	102	51
4	$2\frac{1}{2}$	142	57

These figures clearly suggest that in cases where a comparatively small increase of resistance is required, the most

economical plan is to increase the proportion of tension reinforcement in preference to employing compression reinforcement.

With the object of ascertaining the effect of the two methods of procedure in a case where the resistance has to be largely increased, two further examples will now be taken.

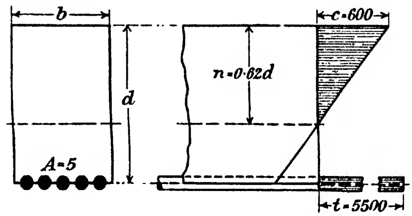


FIG. 50

Example V. Beam as in Example II, but with Greater Excess of Steel in Tension.—In this case the proportion of steel in tension is increased to $A = 5$, as represented in Fig. 50. Owing to the fall of the neutral axis, the steel

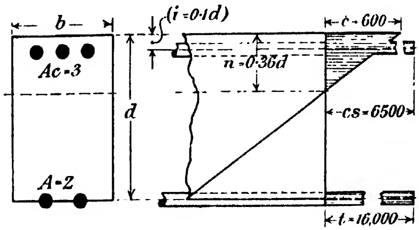


FIG. 51

in tension is employed to great disadvantage and the effect on the resistance moment is that we have the relatively low value $R = 156$.

Example VI. Beam as in Example IV, but with Increased Proportions of Steel in Tension and Compression.—Here, as shown in Fig. 51, we have the same total amount of steel as in Example V, but distributed in a different way, thanks to which the position of the neutral axis permits the working

stresses in the concrete and the tension reinforcement to be more fully utilized. Consequently, for the resistance moment we now get the relative value $R = \cdot 204$, approximately.

From these additional examples, it is evident that compression reinforcement is much more efficient than excess tension reinforcement where considerable increases of resistance are necessary. The only limit to the number and diameter of the bars is that fixed by the dimensions of the beam in which they are to be employed, whereas the limit in the case of excess tension reinforcement is reached at a much earlier stage, being due to the rapid fall of the neutral axis with successive increases of steel.

The chief drawback of compression reinforcement is that as the steel must deform equally with the concrete in which it is embedded, the stress intensity in the steel is governed by the elastic modular ratio of the materials and by the stress intensity in the concrete at a point corresponding with the centre of the reinforcement. Consequently, the stresses developed in the compression reinforcement are very much lower than those in steel employed as tension reinforcement.

CHAPTER VI

THEORY OF REINFORCED CONCRETE

WEB STRESSES IN BEAMS; COMPRESSION AND TENSION MEMBERS; MEMBERS UNDER COMBINED STRESSES

WEB STRESSES IN BEAMS

General Principles.—In the introduction to the general theory of beams at the beginning of Chapter V, reference was made to the distribution of shearing stresses in homogeneous beams, but discussion of the subject in connection with reinforced concrete beams was reserved for the present chapter.

Attention has been given so far to the main horizontal or longitudinal stresses involved in the calculation of resistance moments, which are generally and in many cases quite correctly regarded as denoting the capacity of a beam to carry the required loading.

In the case of reinforced concrete beams, however, the secondary or web stresses may be of such intensity as to cause the failure of the beam before it has been loaded to an extent justified by the calculated resistance moment. Therefore it is necessary to pay careful attention to web stresses in the design of reinforced concrete beams.

The stresses coming under the general head of web stresses include horizontal shearing stress, or bond stress, around the reinforcing bars, shearing stresses in various directions, and tensile and compressive stresses acting in directions not parallel to the axis of the beam.

In resistance moment calculations, it is usually assumed either that there is no tension in the concrete, or that any tension therein is negligible. At sections of a beam where the bending moment is of maximum value, or is sufficient to develop tensile stresses causing virtual, although perhaps not visible, failure of the concrete, the assumption is justified.

But at other sections, where the bending moment is of relatively small value, tension exists in the concrete and cannot be neglected when web stresses are under consideration.

Shearing Stress Around Longitudinal Bars.—In order that true beam action may take place in a reinforced concrete member, there must be an effective web connection between the tension and compression elements, and in beams where web reinforcement is not employed the concrete acts as a web.

As the bending moments vary from point to point in the length of a beam, the amount of tension in the reinforcing bars and the amount of compression in the concrete above the neutral axis vary from section to section along the beam. The increments of tension and compression between successive sections must be connected by means of the web, or, in other words, the increments of tensile stress in the steel must be transferred to, or connected with, the increments of compressive stress in the concrete.

Assuming the absence of web reinforcement, the transference of tension from the longitudinal bars to the surrounding concrete is accompanied by horizontal shearing stress, sometimes termed *grip, bond, or adhesion* stress, tending to destroy the grip or bond between the materials, and to make the bars slip in the concrete. The stress in question is resisted by the grip or bond, which is measured in terms of the surface of the bars in contact with the concrete. Thus the duty of the grip is similar to that of the rivets used in connecting the web members and flanges of a steel lattice girder.

In beams where the bars are straight throughout their length and where the absence of tension in the concrete may be reasonably assumed, the conditions affecting the integrity of the bond between the steel and the concrete are of simple character. But near the ends of beams where some or all of the bars are bent up, the arrangement of the bars and the existence of tensile and compressive stresses in the concrete give rise to complications which vary from case to case.

Vertical and Horizontal Shearing Stresses.—In accordance with accepted theory, vertical and horizontal shearing stresses of varying intensity exist throughout a beam, the intensity of the vertical shearing stress at any point being equal to that of the horizontal shearing stress at the same point.

Horizontal shearing stress in the concrete performs a necessary duty in transmitting increments of tensile stress, transferred from the reinforcing bars, to the compression area of a beam, and in connecting the increments of tensile and compressive stresses on either side of the neutral axis, thereby enabling the concrete to act as the web of the beam.

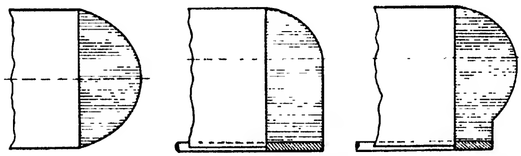


FIG. 52 FIG. 53 FIG. 54

The distribution of shearing force from point to point along a reinforced concrete beam follows the law governing the distribution of the same force in a beam of homogeneous material. But the variation of the intensity of the horizontal and vertical shearing stress at any vertical section differs from that in a homogeneous beam owing to the influence of the reinforcement.

In a homogeneous beam, as already stated, the shearing stress intensity is of maximum value at the neutral axis and diminishes to zero value at the upper and lower extreme fibres, as represented by Fig. 52.

In a reinforced concrete beam, the shearing stress intensity above the neutral axis may be considered, for all practical purposes, to decrease as in a homogeneous beam, although the precise rate of decrease differs with any variation from the straight-line deformation theory. If tension were absent from the concrete, there would be no change in the shearing

stress intensity between the neutral axis and the centre of gravity of the reinforcement. Therefore in accordance with the usual assumptions, Fig. 53 represents the distribution of horizontal shearing stress over any vertical section of a reinforced concrete beam.

As a matter of interest, rather than one of practical importance, it may be mentioned that the existence of longitudinal tension in the concrete has the effect of modifying the distribution of the shearing stress, the intensity of which varies as shown in Fig. 54.

Diagonal Tension.—In addition to the horizontal and vertical shearing stresses hitherto considered, shearing, tensile, and compressive stresses exist in every diagonal direction in a reinforced concrete beam. The horizontal components of these stresses are dealt with in the calculation of the resistance moment, and the diagonal stresses remain to be taken by the web, whether this consists of concrete alone or of concrete with web reinforcement.

It is shown in Chapter V that the direction of the maximum tensile stress in a homogeneous beam is horizontal at the lower fibres and becomes more and more inclined as the neutral axis is approached. Moreover, as will be observed on reference to Fig. 15, the lines of maximum tensile stress towards the ends of a beam are always inclined.

The important difference between the conditions prevailing in the middle portion of a beam and in the two end portions is one deserving attention.

In the middle, where the bending moments are relatively great, the concrete has failed in tension, from the bottom of the beam to some point near the neutral axis, before the average working stress has been reached in the reinforcement. Therefore any tensile resistance on the part of the concrete is of no practical value so far as resistance moment is concerned. The horizontal tensile stresses are much less in the end portions than in the middle portion of a beam, and, in a beam under uniformly distributed loading the shearing stresses are greater at the ends than in the middle. Consequently, the lines of maximum tension are entirely

diagonal in direction and their inclination increases with the preponderance of shearing stress. The tendency of the diagonal tensile stresses in the end portions of a beam is to cause failure of the concrete in tension. The longitudinal bars are not of very much help in preventing such failures, and in beams without properly designed web reinforcement, the main resistance to diagonal tension is that furnished by the tensile strength of the concrete.

Failures by diagonal tension were formerly supposed to result from shear, but their true character is now generally recognized. As the value of the diagonal tensile stresses cannot be determined by calculation, the intensity of the vertical shearing stress is taken as a convenient measure of the diagonal tensile stress developed in a beam, with the understanding that the actual diagonal tensile stress is considerably greater than the calculated shearing stress.

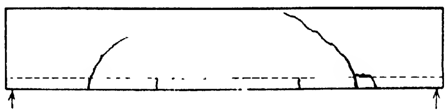


FIG. 55

Fig. 55 illustrates the position and direction of the cracks in a typical diagonal tension failure, the reinforcing bars being straight throughout. Such failures usually start from tension cracks in the bottom of the beam and extend more or less vertically as far as the reinforcement. Above this the cracks run in a diagonal direction as shown.

Diagonal tension failures in beams without web reinforcement generally occur without previous warning. In beams where some of the bars are bent up towards the ends, the increased resistance to diagonal tension is very marked if the bars are securely anchored at their end, and failure only occurs after due warning. If bent-up bars are not effectively anchored, their efficiency in resistance to diagonal tension is relatively small, a point apparently indicating that the bars are liable to slip in the concrete by failure of the bond.

Bent-up bars are frequently employed in connection with U-shaped stirrups, and this combination has been found to give high web resistance and slow failure, with ample warning.

In the case of continuous beams, where the upper part of the beam near the support is in tension, the bars which are bent up on either side of the support should be shaped so that the inclined part may terminate near the points of contraflexure, being continued horizontally over and well beyond the support. Wherever practicable, the bars should be continuous over the supports of two or more spans.

COMPRESSION MEMBERS

Definitions.—As stated at the beginning of Chapter V, the general expression *compression member* denotes all varieties of members subject to longitudinal compression.

A *column* is a vertical support which, in architecture, consists of a cylindrical or slightly tapering *shaft* set vertically upon a *base* and surmounted by a *capital.*

A *pillar* is a vertical support of any form, generally distinguished by architects from a *column* for the reasons that it may be of any cross sectional shape and is not governed by the rules of classic architecture.

A *strut* is a vertical, horizontal, or diagonal brace or support. In engineering, the term *strut* is often employed specifically to denote any compression member in a framed structure.

The principles underlying the design of compression members apply equally to columns, pillars, and struts. In the present section, and elsewhere in this work, the word *column* is used as a convenient term denoting a typical form and not all varieties of compression members.

General Considerations.—For columns and other structural members subject to direct compression alone, concrete is more economical than steel because, using the values generally applicable to comparisons of the kind, the cost of concrete is one-fiftieth that of steel, and its compressive strength is one-thirtieth that of steel.

Therefore, while for equal strength in compression the

sectional area of a concrete column would be thirty times that of a solid steel column, the cost of the much larger quantity of concrete would be only three-fifths that of the steel.

From the foregoing comparison, it is evident that reinforced concrete cannot be so economical as plain concrete in the form of simple compression members, and that the relative cost of reinforced concrete increases in direct proportion to the amount of steel included as compression reinforcement.

In comparing the cost of reinforced concrete and steel in the form of compression members we have to take into consideration not only the amount of steel employed as reinforcement in the concrete but also the fact that, in accordance with the elastic theory, the compressive strength of the reinforcing steel cannot be fully developed without involving excessive stress in the concrete.

Thus, on the basis $m = 15$, compressive stress in the steel must not be more than fifteen times that in the concrete. It follows that the cost of reinforced concrete increases rapidly with the proportion of steel, and that a point is soon reached where the cost of reinforced concrete equals that of steel.

The foregoing comparisons, like those made with regard to beams, are merely intended to illustrate theoretical principles in a general way, and it must not be inferred that the values taken for the relative strength and cost of concrete and steel are of universal application in everyday practice.

In comparing the cost of practical columns of reinforced concrete and steel, respectively, three points deserving attention are the following—

1. The use of concrete possessing high compressive strength may permit the resistance of the reinforcing steel to be utilized nearly up to the permissible limit for steel employed independently.

2. The cost of riveted steelwork, as represented by built-up columns, was about 70 per cent. more than that of ordinary rolled bars at pre-war rates, and will probably be

something like 100 or 150 per cent. more in future unless the increased cost of labour is accompanied by increased production.

3. The cost of encasing steel columns, necessary in building construction for protection against fire, and the cost of painting and maintaining unprotected steel columns in other classes of construction, are items of expenditure which are obviated by the employment of reinforced concrete.

Longitudinal and Transverse Reinforcement.—Although a reinforced concrete column might be made with

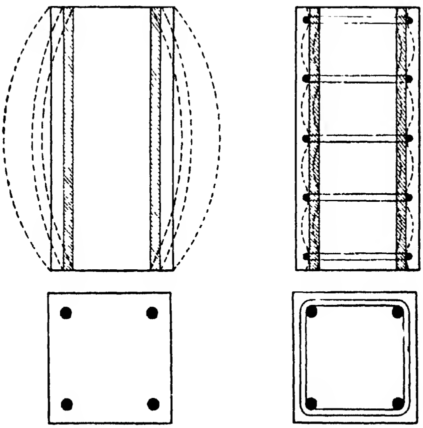


FIG. 56 FIG. 57

longitudinal reinforcement alone, it would represent an incomplete and undesirable type of design.

Transverse reinforcement of some kind is necessary to brace the main bars and to share with them the duty of binding the concrete together laterally.

A good idea of the manner in which longitudinal and transverse reinforcements act may be obtained by considering a very short column, such as that represented in Fig. 56 reinforced by longitudinal bars. The effects of loading applied at the top are to tend to cause longitudinal shortening

and lateral expansion of the concrete, accompanied by outward bending of the bars, as indicated diagrammatically by broken lines. As might be expected, the bars would bend more readily because their length is great in proportion to their diameter.

The bars can be rendered far more effective by the addition of transverse links or ties at frequent intervals, as in Fig. 57, so as to reduce the bending of the bars to a minimum, and to furnish resistance to the lateral expansion of the concrete. As a matter of fact, the bars can only bend between the links to a small extent and at the links to the still smaller extent permitted by stretching of the links in tension. In actual practice the expansion of the concrete and the bending of the bars would be infinitesimal.

By the employment of transverse links, the longitudinal bars are enabled to offer increased resistance to shortening on the part of the concrete, and thereby the tendency of the latter to expand laterally is correspondingly reduced.

The effect of concerted action in the manner briefly outlined is to bring into play that combination of effort which is the basis of reinforced concrete construction.

Longitudinal Reinforcement of Primary Importance.—Whether a compression member is *short* or *long*, in relation to its transverse dimensions, the longitudinal bars always come first in order of importance. For instance, in a short column they are of direct aid to the concrete by helping to withstand compression, and to the transverse reinforcement by reducing the compression and the consequent lateral expansion of the concrete. Again, in a long column, they perform similar duties and also impart stiffness by offering resistance to flexure resulting from what is generally termed *column action.*

Further, in the case of piles, longitudinal reinforcement is absolutely necessary to permit members of this class to be transported and slung before they have been driven. Without such reinforcement, piles would run the risk of undue strain as a result of the bending moments due to preliminary handling.

Transverse Reinforcement of Secondary Importance.—Valuable as transverse reinforcement undoubtedly is, it naturally takes a second place as an auxiliary to longitudinal reinforcement. This fact is here emphasized for the reason that the interesting results obtained by the French Commission du Ciment Armé as to the effects of close spiral binding have sometimes been misinterpreted and misapplied.

The general purport of the results in question may be expressed in the statement that transverse reinforcement in the form of close spiral binding adds about 2·4 times as much strength to the concrete of columns as would be imparted by an equal amount of steel in the form of longitudinal bars.

So far as ultimate strength is concerned, this is doubtless true, but the additional strength cannot be developed without causing undue shortening and excessive lateral deflection. Consequently, this special form of transverse reinforcement cannot be regarded as a substitute for longitudinal reinforcement, particularly within the limits of working stresses.

Tension in Transverse Reinforcement.—If the concrete of a column subjected to axial compression is restrained by transverse reinforcement, the resulting lateral compression tends to neutralize the bulging due to axial compression. Therefore the efficiency of transverse reinforcement depends upon the closeness of its spacing and the tensile resistance of the steel to the lateral expansion of the concrete.

In accordance with the average value of Poisson's ratio, approximately 0·1, for concrete under ordinary working loads, the lateral deformation may be taken at about one-tenth of the longitudinal deformation due to axial compression. On this basis, the tensile stress in the transverse reinforcement is equal to only one-tenth of the compressive stress in the longitudinal reinforcement.

Beyond the elastic limit of the longitudinal reinforcement in compression, the corresponding tensile stress in the transverse reinforcement increases with the axial compression and with the increasing value of Poisson's ratio, which attains a maximum of about 0·25 at the breaking load of the

column in compression. The tensile stress in the transverse reinforcement may then be expected to be approximately equal to the ordinary working stress for steel in tension. The fact that the transverse reinforcement is not overstressed, even when the ultimate strength of the column is reached, explains the efficiency of closely-spaced binding in raising the ultimate strength of reinforced concrete compression members.

"Short" and "Long" Columns.—The distinction drawn in text-books on applied mechanics between "short" columns, which fail by crushing of the material, and "long" columns, which fail either by flexure alone or by flexure and crushing, applies also to reinforced columns and compression members generally. The terms *short* and *long* are by no means appropriate in themselves, as the idea intended to be conveyed is that of shortness or of length relative to the thickness of a column. Therefore, "stout" and "slender" would be more suitable adjectives.

It should be pointed out, however, that the reinforced concrete columns used in building construction and in many classes of engineering construction rarely exceed twelve or fifteen diameters in length, or in unsupported length, and that the results of well authenticated tests indicate little or no difference in strength for ratios of length to least diameter up to 20 or 25. Moreover, according to Professor Talbot, the difference in strength between a column 15 diameters long and one 5 diameters long is less than the variation among several columns of the same length.

Compression members whose unsupported length is more than eighteen times the least diameter may develop what is commonly termed *column action* and must be designed with due regard to the stresses resulting from flexure.

It is, of course, quite impossible to draw any arbitrary dividing line between the two classes of members, as one merges into the other by imperceptible degrees. In the case of members liable to flexure, the longitudinal reinforcement is particularly valuable, and any attempt to minimize the proportion of steel so applied by increasing the

proportion of transverse ties or binding can only result in smaller resistance to flexure.

Columns Under Eccentric Loading.—In reinforced concrete columns, as generally employed in ordinary building construction, the ratio of unsupported length to least diameter is small and *column action* does not appear to take place. If the loading is concentric, it is only necessary to take compressive stresses into account, and the same applies to columns where the eccentricity of the loading is relatively small. In the latter case, the effect of the loading may simply be to make the compressive stress at one side greater than that at the opposite side of the column.

The designer must on no account take it for granted that this will be the only result of eccentric loading, and must carefully inquire into the actual conditions in every case. If, owing to relatively great eccentricity of loading, or to any equivalent cause, the compressive stress at either side of a column is replaced by tensile stress, the resistance of the member will be greatly reduced.

This remark applies with special force to columns where the ratio of unsupported length to least diameter is such that column action, involving the development of tensile stress, may be expected even in the case of concentric loading.

Therefore any column which may be called upon to withstand tension resulting from eccentricity of loading must be calculated as a member subject to combined compressive and bending stresses.

TENSION MEMBERS

In view of the distinctive characteristics of reinforced concrete, it is fairly obvious that from the theoretical standpoint, concrete and steel cannot be combined efficiently in the form of members subject to direct tension.

Nevertheless, concrete and steel can occasionally be employed with practical advantage in the construction of members, resembling compression members in exterior form and in the arrangement of the steel bars, and intended for resistance to direct tension. For instance, in a framed

structure the ties are preferably made of reinforced concrete in order that the harmony of the design may be preserved, and that the steel may be protected from the corrosion which would result if naked steel tie bars were employed.

Again, in braced girders subject to variable load conditions, compression web members may be called upon to withstand tension, and tension web members to withstand compression; or where live loads are in question, some of the web members may have to withstand either tension or compression.

Hence, concrete and steel are sometimes usefully employed in the construction of what may be termed, by courtesy more than anything else, "reinforced concrete tension members."

It would be quite possible to design tension members so as to provide for the co-operation of concrete and steel in joint resistance to stress in proportions governed by their respective elastic properties, and the combination would then be justly entitled to the designation "reinforced concrete." Any design complying with such a condition would necessarily involve very low and uneconomical working stresses for the steel, as concrete of average quality virtually fails in tension when its total elongation corresponds with a stress of about 5,000 lb. per square inch in the steel.

Even assuming the employment of steel in a tension member and the development of an economical working stress, the virtual failure of the concrete in tension does not involve its destruction, because, as previously shown, the extensibility of the concrete is very greatly developed by the steel, and although its tensile resistance may be destroyed, the material retains its usefulness in other respects.

MEMBERS UNDER COMBINED DIRECT AND BENDING STRESSES

With comparatively rare exceptions, the direct stress in reinforced concrete work is compressive. For this reason we need not discuss the effects of combined tensile and bending stresses, particularly as the principles involved are

similar to those applying to combined compressive and bending stresses.

In any member subject to the last-named combination, the resultant fibre stress may be either (1) wholly compressive, or (2) compressive on one side and tensile on the

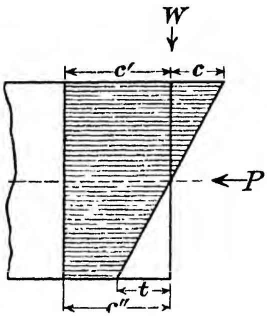


FIG. 58

other side. The dividing line between these two conditions is represented by the case where on one side there is compressive stress, and on the other side the compressive stress due to direct pressure is neutralized by the tensile stress due to bending moment.

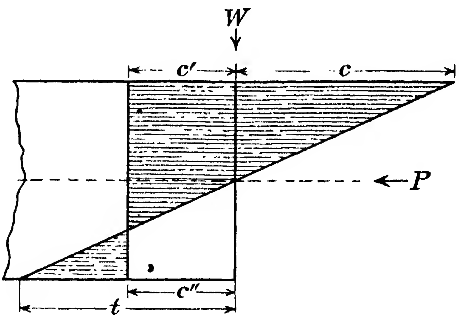


FIG. 59

The three sets of conditions are illustrated by Figs. 58, 59, and 60, these diagrams applying equally to either

struts or beams so long as the external forces W and P act in the relative directions indicated.

In Fig. 58 let the compressive and tensile stresses due to bending moment be $c = 1$ and $t = 1$, and let the compressive stresses due to direct pressure be $c' = 2$ and $c'' = 2$. Then the fibre stresses will be wholly compressive, and their relative values will be—

At the top . $(c + c') = (1 + 2) = 3$
At the bottom . $(c'' - t) = (2 - 1) = 1$

Similarly, in Fig. 59, for $c = 4$, $t = 4$, $c' = 2$, and $c'' = 2$, we have compressive fibre stress at the top and tensile fibre stress at the bottom, the relative values of the stresses being—

At the top . $(c + c') = (4 + 2) = 6$
At the bottom . $(t - c'') = (4 - 2) = 2$

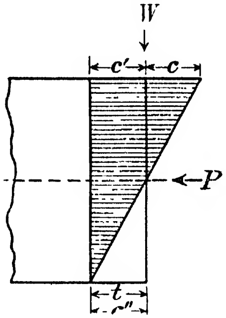


FIG. 60

Finally, as shown in Fig. 60, for $c = 1$, $t = 1$, $c' = 1$, and $c'' = 1$, we have compressive stress at the top and no stress at the bottom. Thus—

At the top . $(c + c') = (1 + 1) = 2$
At the bottom . $(t - c'') = (1 - 1) = 0$

The case represented by Fig. 60 is virtually equivalent to that shown in Fig. 58 so far as practical work is concerned because the stress between the top and bottom fibres is wholly compressive. It is dealt with here simply for the purpose of illustrating general principles.

resistance moment in terms of the tensile stress by the symbol Rt, we have—

$$Rt = T{\cdot}a$$
$$= (t{\cdot}r{\cdot}b{\cdot}d)\left(1 - \frac{n_{\prime}}{3}\right)d$$

By re-grouping these factors and bringing the symbols of the pure ratios in front of the symbols of magnitudes, we can obtain an equation in standard form.

The dominant ratio r, from which most of the other ratios can be obtained, appropriately occupies the first place. Symbols denoting pure ratios are naturally followed by the symbols of the magnitudes in the order, (1) force, (2) space.

The resultant equation therefore takes the following standard form—

$$Rt = r\left(1 - \frac{n_{\prime}}{3}\right) t{\cdot}b{\cdot}d^2$$

If we separate the geometrical magnitudes, or those which refer solely to the size of the beam, the equation becomes

$$Rt = \left[r\left(1 - \frac{n_{\prime}}{3}\right)t\right] b{\cdot}d^2 \quad . \quad . \quad . \quad (1)$$

Resistance Moment in Compression.—Taking moments about the centre of tension in the steel and denoting the resistance moment in terms of the compressive stress by the symbol Rc, we have

$$Rc = C{\cdot}a$$
$$= \left(n_{\prime}{\cdot}\tfrac{1}{2}\, c{\cdot}b{\cdot}d.\right)\left(1 - \frac{n_{\prime}}{3}\right)d$$

By bringing similar quantities together and arranging them as before, we obtain the standard form of equation

$$Rc = n_{\prime}\left(1 - \frac{n_{\prime}}{3}\right) \tfrac{1}{2}c{\cdot}b{\cdot}d^2$$

Separating, as before, the magnitudes which refer solely to the size of the beam, we have

$$Rc = \left[n_{\prime}\left(1 - \frac{n_{\prime}}{3}\right) \tfrac{1}{2}c\right] b{\cdot}d^2 \quad . \quad . \quad . \quad (2)$$

Formulæ for Stresses and Ratios.—In accordance with assumption (3), denoting tensile strain in the steel by the

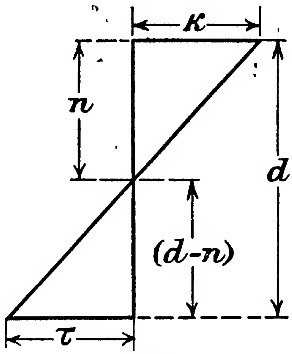


FIG. 63

symbol τ, and compressive strain in the concrete by κ, we have the relations expressed in the following equations, the symbols employed being shown in Fig. 63.

$$\frac{\kappa}{n} = \frac{\tau}{d-n}$$

$$\frac{\kappa}{\tau} = \frac{n}{d-n}$$

But $n = n_{\prime} \cdot d$, and $(d-n) = d - n_{\prime} \cdot d = (1 - n_{\prime})d$

$$\therefore \quad \frac{\kappa}{\tau} = \frac{n_{\prime} \cdot d}{(1 - n_{\prime})d}$$

Cancelling d from the right-hand term of this equation we obtain

$$\frac{\kappa}{\tau} = \frac{n_{\prime}}{(1 - n_{\prime})}$$

As $Es = \frac{t}{\tau}$ and $Ec = \frac{c}{\kappa}$ we have

$$\tau = \frac{t}{Es} \text{ and } \kappa = \frac{c}{Ec}$$

Whence we derive the equation

$$\frac{\kappa}{\tau} = \frac{c/Ec}{t/Es}$$

If we multiply both the numerator and the denominator of the right-hand term of the equation by Es we get

$$\frac{\kappa}{\tau} = \frac{Es \cdot c/Ec}{Es \cdot t/Es}$$

and by cancellation

$$\frac{\kappa}{\tau} = \frac{Es \cdot c/Ec}{t}$$

Dividing the numerator and the denominator of the right-hand term by t we obtain

$$\frac{\kappa}{\tau} = \frac{Es \cdot c/Ec \cdot t}{t/t}$$

and by cancellation

$$\frac{\kappa}{\tau} = Es \cdot c/Ec \cdot t$$

$$= \frac{Es \cdot c}{Ec \cdot t}$$

This may be written

$$\frac{\kappa}{\tau} = \frac{Es}{Ec} \cdot \frac{c}{t}$$

The ratio of the elastic modulus of the steel to the elastic modulus of the concrete (Es/Ec) is termed the *modular ratio.* Denoting this ratio by the symbol m we have

$$\frac{\kappa}{\tau} = m \cdot \frac{c}{t}$$

By a previous equation it is shown that

$$\frac{\kappa}{\tau} = \frac{n_1}{(1 - n_1)}$$

Therefore, by combining the two equations, we get

$$m \cdot \frac{c}{t} = \frac{n_1}{(1 - n_1)}$$

Whence

$$n_1 = \frac{1}{1 + (t/m \cdot c)} \quad . \quad . \quad . \quad . \quad (3)$$

$$t = \frac{(1 - n_1)}{n_1} \cdot m \cdot c \quad . \quad . \quad . \quad . \quad (4)$$

$$c = \frac{n_1}{(1 - n_1)} \cdot \frac{t}{m} \quad . \quad . \quad . \quad . \quad (5)$$

Equating the forces acting above and below the neutral axis, we have

$$T = C$$

$$t \cdot r \cdot b \cdot d = \tfrac{1}{2} c \cdot b \cdot n$$

But $n = n_1 \cdot d$. Therefore we have

$$t \cdot r \cdot b \cdot d = \tfrac{1}{2} c \cdot b \cdot n_1 \cdot d$$

Dividing both sides of this equation by $b \cdot d$, it is reduced to

$$t \cdot r = \tfrac{1}{2} c \cdot n_1$$

or

$$r \cdot t = \tfrac{1}{2} n_1 \cdot c$$

Substituting in this reduced equation, the value for c given in (5), we have

$$r \cdot t = \tfrac{1}{2} n_1 \cdot \frac{n_1}{(1 - n_1)} \cdot \frac{t}{m}$$

Then, combining like terms and removing t from both sides, we obtain the equation

$$r = \frac{n_1^2}{2(1 - n_1)m} \quad . \quad . \quad . \quad . \quad (6)$$

From (6) we obtain

$$n_1^2 = 2(1 - n_1)m \cdot r$$

This is an adfected quadratic equation having two roots. But since we know that n_1, m and r are all positive numbers there can only be one rational solution. We may therefore proceed in the following manner—

First, expand the terms within the brackets and obtain

$$n_1^2 = 2m \cdot r - 2m \cdot r \cdot n_1$$

Then, add $2m\cdot r\cdot n_1$ to both sides and cancel.

Thus,

$$n_1^2 + 2m\cdot r\cdot n_1 = 2m\cdot r$$

The coefficient of the second term is $2m\cdot r$.

We must complete the broken square by adding the square of half the coefficient of the second term $(m\cdot r)^2$ to both sides of the equation.

This will give

$$n_1^2 + 2m\cdot r\cdot n_1 + (m\cdot r)^2 = (m\cdot r)^2 + 2m\cdot r$$

But

$$n_1^2 + 2m\cdot r\cdot n_1 + (m\cdot r)^2 = (n_1 + m\cdot r)^2$$

Therefore we may write

$$(n_1 + m\cdot r)^2 = (m\cdot r)^2 + 2m\cdot r$$

By extracting the square root of both sides we obtain

$$n_1 + m\cdot r = \sqrt{([m\cdot r]^2 + 2m\cdot r)}$$

Subtracting $m\cdot r$ from both sides, we obtain

$$n_1 = \sqrt{([m\cdot r]^2 + 2m\cdot r)} - m\cdot r \quad . \quad . \quad . \quad (7)$$

The value of r given by (6) is the "economic" ratio of reinforcement.

If the ratio be lower than that indicated by formula (6) the beam will be stronger in compression than in tension. This means that the compressive resistance of the concrete cannot be fully utilized without exceeding the permissible tensile stress in the reinforcement.

On the other hand, if the ratio be higher than that indicated by the same formula, the beam will be stronger in tension than in compression. This means that the tensile resistance of the steel cannot be fully utilized without exceeding the permissible compressive stress in the concrete.

T-BEAMS WITH SINGLE REINFORCEMENT

Beams of Class I where the Neutral Axis is not below the Under Surface of the Flange.—Formulæ for beams of this class are essentially similar to those for rectangular beams with single reinforcement, the only difference being that a distinction is necessarily made between b, the breadth of the flange, and br, the breadth of the rib.

Basis of the Ratio of Reinforcement.—The ratio of reinforcement for tee beams can be taken either (1) in respect of the area $br \cdot d$, corresponding with the area $b \cdot d$ in the case of rectangular beams, or (2) in respect of the area $b \cdot d$, In the first method, the value of r, as given by formula (13), is variable with the flange breadth, being expressed in terms of a constant area $br \cdot d$, and the value for a flange of unit breadth, $b/br = 1$, is a *basic* value from which, as r increases proportionately with the flange breadth, the ratio of steel for any value of b/br can be readily obtained by multiplication. Therefore this method facilitates direct comparison between values of r for beams of rectangular and T-shaped cross-section. In the second method, the value of r, as given by formula (13a), is constant, being expressed in terms of a variable area $b \cdot d$, but the actual area of steel is the same as that obtained by the first method, because in calculating the steel area the smaller value of r is multiplied by a larger area, $b \cdot d$, and similarly, the calculated resistance moment is the same as that obtained by the first method. Formulæ for both methods of procedure are given in this section.

Formulæ for Resistance Moment.—The derivation of the subjoined equations is sufficiently explained by the article dealing with formulæ for rectangular beams, and by Figs. 64 and 65.

Taking moments about the centre of compression in the concrete

$$Rt = \left[r\left(1 - \frac{n_t}{3}\right)t\right] br \cdot d^2 \qquad (8)$$

The ratio $r = A/br \cdot d$ for use in this equation can be calculated by formula (13), where the factor b/br provides for variation of the ratio of steel proportionately with any variation in the relative breadths of the flange.

A modification of formula (8) is

$$Rt = \left[r\left(1 - \frac{n_t}{3}\right)t\right] b \cdot d^2 \quad . \quad . \quad . \quad . \quad (8a)$$

Here, b = breadth of flange, and the value of r must be

expressed in terms of the area $b \cdot d$, and can be calculated by (13a), which is identical with (6) for rectangular beams.

Taking moments about the centre of tension in the steel

$$Rc = \left[n_{\prime}\left(1 - \frac{n_{\prime}}{3}\right) \cdot \tfrac{1}{2}c\right] b \cdot d^2 \quad . \quad . \quad . \quad . \quad (9)$$

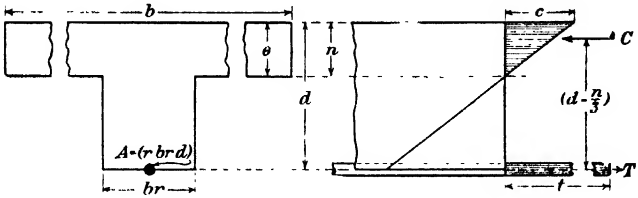


FIG. 64 FIG. 65

Formulæ for Stresses and Ratios.—The derivation of equations (10) to (22) is similar to that of equations (3) to (7).

$$n_{\prime} = \frac{1}{1 + t/m \cdot c} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$t = \frac{(1 - n_{\prime})}{n_{\prime}} \cdot m \cdot c \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$c = \frac{n_{\prime}}{(1 - n_{\prime})} \cdot \frac{t}{m} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$r = \frac{n_{\prime}^2}{2(1 - n_{\prime})m} \cdot \frac{b}{br} \quad . \quad . \quad . \quad . \quad . \quad (13) *$$

$$r = \frac{n_{\prime}^2}{2(1 - n_{\prime})m} \quad . \quad . \quad . \quad . \quad . \quad (13a)\dagger$$

$$n_{\prime} = \sqrt{([m \cdot r \cdot br/b]^2 + [2m \cdot r \cdot br/b]) - [m \cdot r \cdot br/b]} \quad . \ (14) *$$

$$n_{\prime} = \sqrt{([m \cdot r]^2 + [2m \cdot r]) - m \cdot r} \quad . \quad . \quad . \quad (14a)\dagger$$

Beams of Class II where the Neutral Axis is below the Under Surface of the Flange.—For T-beams of this class (see Figs. 66 and 67) some of the formulæ are not

* For use in (8). † For use in (8a).

quite the same as those for beams of Class I, because the centre of compression in the concrete is at the centroid of a trapezoidal instead of a triangular stress area (compare Figs. 65 and 67).

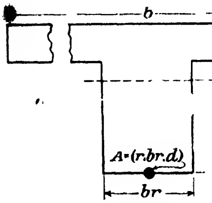


FIG. 66

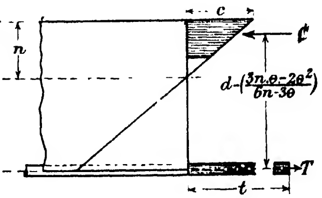


FIG. 67

Thus, by the usual rule for the centroid of a trapezoid, the depth of the centre of compression in a beam of Class II is

$$\left(\frac{3n_1\cdot\theta_1 - 2\theta_1{}^2}{6n_1 - 3\theta_1}\right)d$$

Consequently, the arm of leverage of the internal forces is

$$\left(1 - \frac{3n_1\cdot\theta_1 - 2\theta_1{}^2}{6n_1 - 3\theta_1{}^2}\right)d$$

and the mean compressive stress in the concrete is

$$\left(1 - \frac{\theta_1}{2n_1}\right)c$$

Formulæ for Resistance Moment.—In the following equations we neglect the compressive resistance of the small area of concrete in the web, between the under surface of the flange and the neutral axis. (See Fig. 67.)

Taking moments about the centre of compression in the concrete

$$Rt = \left[r\left(1 - \frac{3n_1\cdot\theta_1 - 2\theta_1{}^2}{6n_1 - 3\theta_1}\right)t\right]br\cdot d^2 \quad . \quad . \quad . \quad (15)$$

or modifying the equation in the same way as (8).

$$Rt = \left[r\left(1 - \frac{3n_1\cdot\theta_1 - 2\theta_1{}^2}{6n_1 - 3\theta_1}\right)t\right]b\cdot d^2 \quad . \quad . \quad . \quad (15a)$$

Values of r for (15) must be in terms of $br \cdot d$ and can be calculated by (20). Values of r for (15a) must be in terms of $b \cdot d$ and can be calculated by (20a).

Taking moments about the centre of tension in the steel

$$Rc = \left[\theta_{,} \left(1 - \frac{3n_{,} \cdot \theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c \right] b \cdot d^{2} \quad . \ (16)$$

Formulæ for Stresses and Ratios.—Equations (17) to (19) are the same as the corresponding equations for rectangular beams and T-beams of Class I.

Equations (20) and (21) are essentially similar to those for the same classes of beams, a point which may be demonstrated by reducing the ratio b/br to unity and taking $\theta_{,} = n_{,}$.

$$n_{,} = \frac{1}{1 + t/m \cdot c} \quad . \quad . \quad . \quad . \quad . \ (17)$$

$$t = \frac{(1 - n_{,})}{n_{,}} \cdot m \cdot c \quad . \quad . \quad . \quad . \quad . \ (18)$$

$$c = \frac{n_{,}}{(1 - n_{,})} \cdot \frac{t}{m} \quad . \quad . \quad . \quad . \quad . \ (19)$$

$$r = \frac{n_{,} \cdot \theta_{,}(1 - \theta_{,}/2n_{,})}{(1 - n_{,})m} \cdot \frac{b}{br} \quad . \quad . \quad . \ (20)^{*}$$

$$r = \frac{n_{,} \cdot \theta_{,}(1 - \theta_{,}/2n_{,})}{(1 - n_{,})m} \quad . \quad . \quad . \ (20a)^{\dagger}$$

$$n_{,} = \frac{2m \cdot r + \theta_{,}^{2} \cdot (b/br)}{2m \cdot r + 2\theta_{,}(b/br)} \quad . \quad . \quad . \ (21)^{*}$$

$$n_{,} = (2m \cdot r + \theta_{,}^{2})/(2m \cdot r + 2\theta_{,}) \quad . \quad . \ (21a)^{\dagger}$$

RECTANGULAR BEAMS WITH DOUBLE REINFORCEMENT

Formulæ for Resistance Moment.—Let Fig. 68 represent the cross-section of a rectangular beam where the tension reinforcement consists of a round steel bar placed at a suitable distance from the lower surface and the compression

* (1) For use in (15). † For use in (15a).

reinforcement consists of a round steel bar placed at a suitable distance from the upper surface of the concrete.

The stress in the reinforcement may be regarded as being of uniform intensity and as acting at the centre of each bar.

In cases where two or more bars are employed as tension reinforcement, or as compression reinforcement, the stress is regarded as acting at the centre of gravity of each group of bars.

In accordance with assumption (7), nothing is allowed for the tensile resistance of the concrete, and tension reinforcement must be proportioned to withstand the total tension due to bending moment.

In the area above the neutral axis compressive stress is resisted partly by the concrete and party by the compression reinforcement, the latter being inset at any distance, i, below the top of the beam.

Let Fig. 69 be the stress diagram for the section of the beam represented in Fig. 68.

Then as t = tensile stress intensity in the steel and $r{\cdot}b{\cdot}d$ = sectional area of steel, the total tension is $t(r{\cdot}b{\cdot}d) = t{\cdot}r{\cdot}b{\cdot}d$.

Similarly, as cs = compressive stress intensity in the steel and $rc{\cdot}b{\cdot}d$ = sectional area of steel the amount of the total compression taken by the steel is $cs(rc{\cdot}b{\cdot}d) = cs{\cdot}rc{\cdot}b{\cdot}d$.

Compressive stresses in the concrete above the neutral axis, regarded as acting at the centroid of the triangular stress area, are resisted by the concrete in the area $b{\cdot}n = b{\cdot}n_{\prime}{\cdot}d$ and the mean compressive stress in the concrete throughout the area $b{\cdot}n_{\prime}{\cdot}d$ is $\frac{1}{2}c$.

Consequently, the total compressive force above the neutral axis of the beam is made up of two elements, one acting at the centre of the reinforcement and the other acting at the centroid of the triangular stress area in the concrete. Thus the total compression is

$$cs{\cdot}rc{\cdot}b{\cdot}d + \tfrac{1}{2}c{\cdot}b{\cdot}n_{\prime}{\cdot}d = (cs{\cdot}rc + \tfrac{1}{2}c{\cdot}n_{\prime})b{\cdot}d$$

The two forces $t{\cdot}r{\cdot}b{\cdot}d$ and $(cs{\cdot}rc + \frac{1}{2}c{\cdot}n_{\prime})b{\cdot}d$ act in opposite directions, as shown by arrows in Fig. 69, and to comply with the conditions of equilibrium they must be equal.

The resistance moment of the beam, which must be equal to the bending moment due to the total load on the beam, including the applied load and the weight of the materials, is determined by taking moments about the centres of compression in the concrete and the steel, and about the

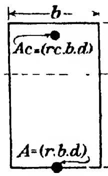


FIG. 68

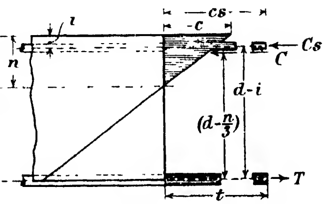


FIG. 69

centre of tension in the steel, the forces being those stated above and the arms of the couples $(d - \frac{1}{3}n)$ and $(d - i)$, as represented in Fig. 69.

Whence we obtain

Taking moments about the centre of compression in the concrete

$$Rt = \left[r\left(1 - \frac{n_1}{3}\right)t + rc\left(\frac{n_1}{3} - i_1\right)cs\right] b\cdot d^2. \qquad (22)$$

Taking moments about the centre of compression in the steel

$$Rt = \left[r\left(1 - i_1\right)t - n_1\left(\frac{n_1}{3} - i_1\right)\tfrac{1}{2}c\right] b\cdot d^2. \qquad (23)$$

Taking moments about the centre of tension in the steel

$$Rc = \left[n_1\left(1 - \frac{n_1}{3}\right)\tfrac{1}{2}c + rc\left(1 - i_1\right)cs\right] b\cdot d^2 \qquad (24)$$

In beams where the compression reinforcement is placed so that the centroid of the area of the steel bars coincides with the centre of compression in the concrete, making $i_1 = n_1/3$, formulæ (22) and (23) can be simplified as follows—

$$Rt = \left[r\left(1 - \frac{n_1}{3}\right)t\right] b\cdot d^2 \qquad (25)$$

and formula (24) becomes,

$$Rc = \left[\left(n_{\prime}\cdot\tfrac{1}{2}c + rc\cdot cs\right)\left(1 - \frac{n_{\prime}}{3}\right)\right]b\cdot d^2 \quad . \quad . \quad (26)$$

The reason for the omission of the second term within the brackets in (22), (23), and (24), is that its value becomes 0 when $i = n/3$ and when $i_{\prime} = n_{\prime}/3$.

The latter condition is frequently convenient because in addition to simplifying calculations it generally ensures an adequate thickness of concrete above the compression reinforcement.

Cases sometimes occur, however, where the compression reinforcement may be placed with advantage either at a greater or a smaller distance above the neutral axis. For the design of such beams the unabridged equations (22), (23), and (42) are necessary.

Formulæ for Stresses and Ratios.—For beams where the relations between r and rc have been determined by formulæ (32) and (33) we have, as before,

$$n_{\prime} = \frac{1}{1 + t/m\cdot c} \quad . \quad . \quad . \quad . \quad (27)$$

$$t = \frac{(1 - n_{\prime})}{n_{\prime}}\cdot m\cdot c \quad . \quad . \quad . \quad . \quad (28)$$

$$c = \frac{n_{\prime}}{(1 - n_{\prime})}\cdot\frac{t}{m} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

For the compressive stress intensity in the steel above the neutral axis, we have

$$cs = \frac{(n_{\prime} - i_{\prime})}{n_{\prime}}\cdot m\cdot c \quad . \quad . \quad . \quad . \quad (30)$$

$$cs = \frac{(n_{\prime} - i_{\prime})}{(1 - n_{\prime})}\cdot t \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Equating the forces acting on either side of the neutral axis we have

$$t\cdot r\cdot b\cdot d = (\tfrac{1}{2}c\cdot n_{\prime} + cs\cdot rc)b\cdot d$$

and by reduction

$$t\cdot r = (\tfrac{1}{2}c\cdot n_{\prime} + cs\cdot rc)$$

Substituting the values for c and cs given by (29) and (31) and reducing, we obtain

$$r = \frac{n_1^2}{2(1-n_1)m} + \frac{(n_1 - i_1)}{(1-n_1)} \cdot rc \quad . \quad . \quad . \quad . \quad (32)$$

$$rc = \frac{2(1-n_1)m \cdot r - n_1^2}{2(n_1 - i_1)m} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$n_1 = \sqrt{([m(r + rc)]^2 + 2m[r + rc \cdot i_1]) - m[r + rc]} \quad . \quad (34)$$

It should be noted that in all beams where the tension and compression reinforcement is employed in the proportions governed by (32) and (33), the value for n_1 as determined by (34) is the same as that given by (27).

T-BEAMS WITH DOUBLE REINFORCEMENT

Beams of Class I where the Neutral Axis is not below the Under Surface of the Flange.—Formulæ for beams of this class are essentially similar to those for rectangular beams with double reinforcement, the only difference being that in some of them a distinction is necessarily made between b, the breadth of the flange, and br, the breadth of the rib.

Basis of the Ratio of Reinforcement.—As previously explained, the ratio of reinforcement for tee beams can be taken either (1) in respect of the area $br \cdot d$, corresponding with the area $b \cdot d$ in the case of rectangular beams, or (2), in respect of the area $b \cdot d$. Formulæ for both methods of procedure are included in this section.

Formulæ for Resistance Moment.—The derivation of the subjoined equations is sufficiently explained by preceding articles, and by Figs. 70 and 71.

Taking moments about the centre of compression in the concrete

$$Rt = \left[r\left(1 - \frac{n_1}{3}\right)t + rc\left(\frac{n_1}{3} - i_1\right)cs \right] br \cdot d^2 \quad . \quad . \quad . \quad (35)$$

or, modifying the equation for use with values of r and rc, expressed in terms of the area $b{\cdot}d$., and in accordance with (45a) and (46a),

$$Rt = \left[r\left(1 - \frac{n_1}{3}\right)t + rc\left(\frac{n_1}{3} - i_1\right)cs\right]b{\cdot}d^2 \quad . \quad . \quad . \quad (35a)$$

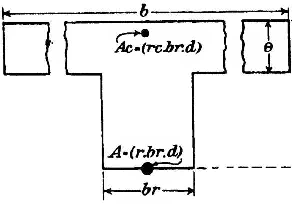


FIG. 70

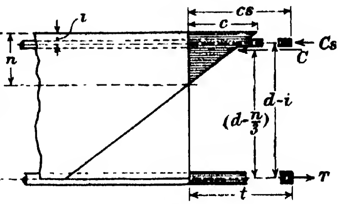


FIG. 71

Taking moments about the centre of compression in the steel

$$Rt = \left[r\left(1 - i_1\right)t - n_1\left(\frac{n_1}{3} - i_1\right)\tfrac{1}{2}c{\cdot}b/br\right]br{\cdot}d^2 \quad (36)$$

or, modifying the equation for use with values of r in terms of the area $b{\cdot}d$,

$$Rt = \left[r\left(1 - i_1\right)t - n_1\left(\frac{n_1}{3} - i_1\right)\tfrac{1}{2}c\right]b{\cdot}d^2 \quad . \quad (36a)$$

Taking moments about the centre of tension in the steel

$$Rc = \left[n_1\left(1 - \frac{n_1}{3}\right)\tfrac{1}{2}c{\cdot}b/br + rc\left(1 - i_1\right)cs\right]br{\cdot}d^2 \quad (37)$$

or, modifying the equation for use with values of rc in terms of the area $b{\cdot}d$,

$$Rc = \left[n_1\left(1 - \frac{n_1}{3}\right)\tfrac{1}{2}c + rc\left(1 - i_1\right)cs\right]b{\cdot}d^2 \quad . \quad (37a)$$

In beams where the compression reinforcement is placed so that the centroid of the area of the steel bars coincides

with the centre of compression in the concrete, or otherwise stated so that $i = n/3$ and $i_{,} = n_{,}/3$ formulæ (35) and (36) can be simplified as follows—

$$Rt = \left[r\left(1 - \frac{n_{,}}{3}\right)t\right] br \cdot d^2 \quad . \quad . \quad . \quad (38)$$

and formula (37) becomes

$$Rc = \left[\left(n_{,} \cdot \tfrac{1}{2}c \cdot b/br + rc \cdot cs\right)\left(1 - \frac{n_{,}}{3}\right)\right] br \cdot d^2 \quad (39)$$

By modification as before, (38) and (39) can be written for use with values of r and rc in terms of the area $b \cdot d$,

$$Rt = \left[r\left(1 - \frac{n_{,}}{3}\right)t\right] b \cdot d^2 \quad . \quad . \quad . \quad (38a)$$

$$Rc = \left[\left(n_{,} \cdot \tfrac{1}{2}c + rc \cdot cs\right)\left(1 - \frac{n_{,}}{3}\right)\right] b \cdot d^2 \quad . \quad (39a)$$

Formulæ for Stresses and Ratios.—The derivation of equations (40) to (47) is similar to that for equations (3) to (7).

$$n_{,} = \frac{1}{1 + t/m \cdot c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$t = \frac{(1 - n_{,})}{n_{,}} \cdot m \cdot c \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$c = \frac{n_{,}}{(1 - n_{,})} \cdot \frac{t}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

$$cs = \frac{(n_{,} - i_{,})}{n_{,}} \cdot m \cdot c \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$cs = \frac{(n_{,} - i_{,})}{(1 - n_{,}{}^2)} \cdot t \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$r = \frac{n_{,}{}^2}{2(1 - n_{,})m} \cdot \frac{b}{br} + \frac{(n_{,} - i_{,})}{(1 - n_{,})} \cdot rc. \quad . \quad (45)^*$$

* For use in (35), (36), (37), (38), and (39).

$$r = \frac{n_1^2}{2(1-n_1)m} + \frac{(n_1 - i_1)}{(1-n_1)} \cdot rc \quad . \quad . \quad . \quad (45a)\dagger$$

$$rc = \frac{2(1-n_1)m \cdot r - n_1 \cdot b/br}{2(n_1 - i_1)m} \quad . \quad . \quad . \quad (46)^*$$

$$rc = \frac{2(1-n_1)m \cdot r - n_1^2}{2(n_1 - i_1)m} \quad . \quad . \quad . \quad (46a)\dagger$$

$$n_1 = \sqrt{\left(\left[\frac{m(r+rc)br}{b}\right]^2 + \frac{2m(r+rc \cdot i_1)br}{b}\right)} - \frac{m(r+rc)br}{b} \quad (47)^*$$

$$n_1 = \sqrt{([m(r+rc)]^2 + 2m[r+rc \cdot i_1])} - m[r+rc] \quad . \quad (47a)\dagger$$

Beams of Class II where the Neutral Axis is below the Under Surface of the Flange.—For reasons previously stated, some of the formulæ for beams of this class differ from the corresponding formulæ for beams of Class I.

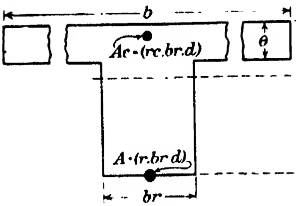


FIG. 72

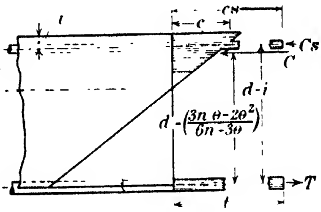


FIG. 73

Formulæ for Resistance Moment.—In the following equations we neglect the compressive resistance of the small area of concrete in the web, between the under surface of the flange and the neutral axis. (See Figs. 72 and 73.)

Taking moments about the centre of compression in the concrete

$$Rt = \left[r\left(1 - \frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)t + rc\left(\frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1} - i_1\right)cs\right]br \cdot d^2 \quad (48)$$

or, using values of r and rc in terms of the area $b \cdot d$,

$$Rt = \left[r\left(1 - \frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)t + rc\left(\frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1} - i_1\right)cs\right]b \cdot d^2 \quad (48a)$$

* For use in (35) to (39). † For use in (35a) to (39a).

Taking moments about the centre of compression in the steel

$$Rt = \left[r\left(1 - i_{,}\right)t + \theta_{,}\left(\frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}} - i_{,}\right)\left(\frac{1 - \theta_{,}}{2n_{,}}\right)c\cdot b/br\right]br\cdot d^{2} \quad (49)$$

or, using values of r in terms of the area $b\cdot d$,

$$Rt = \left[r\left(1 - i_{,}\right)t - \theta_{,}\left(\frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}} - i_{,}\right)\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c\right]b\cdot d^{2} \quad (49a)$$

Taking moments about the centre of tension in the steel

$$Rc = \left[\theta_{,}\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c\cdot b/br + rc\left(1 - i_{,}\right)cs\right]br\cdot d^{2} \quad (50)$$

or, using values of rc in terms of the area $b\cdot d$

$$Rc = \left[\theta_{,}\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c + rc\left(1 - i_{,}\right)cs\right]b\cdot d^{2} \quad (50a)$$

In beams where the compression reinforcement is placed so that the centroid of the area of the steel bars coincides with the centre of compression in the concrete, or so that

$$i_{,} = \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}},$$

formulæ (48) and (49) can be simplified as follows—

$$Rt = \left[r\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)t\right]br\cdot d^{2} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and formula (50) becomes

$$Rc = \left[\left\{\theta_{,}\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c\cdot b/br + rc\cdot cs\right\}\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)\right]br\cdot d^{2} \quad (52)$$

Using modified values of r and rc in terms of the area $b\cdot d$, formulæ (51) and (52) can be written—

$$Rt = \left[r\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)t\right]b\cdot d^{2} \quad . \quad . \quad . \quad . \quad . \quad (51a)$$

$$Rc = \left[\left\{\theta_{,}\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c + rc\cdot cs\right\}\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^{2}}{6n_{,} - 3\theta_{,}}\right)\right]b\cdot d^{2} \quad . \quad (52a)$$

Formulæ for Stresses and Ratios.—The derivation of equations (53) to (60) is akin to that for equations (3) to (7).

$$n_{,} = \frac{1}{1 + t/m{\cdot}c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

$$t = \frac{(1 - n_{,})}{n_{,}}{\cdot}m{\cdot}c \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$c = \frac{n_{,}}{(1 - n_{,})}{\cdot}\frac{t}{m} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$cs = \frac{(n_{,} - i_{,})}{n_{,}}{\cdot}m{\cdot}c \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$cs = \frac{(n_{,} - i_{,})}{(1 - n_{,})}{\cdot}t \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$r = \frac{n_{,}{\cdot}\theta_{,}(1 - \theta_{,}/2n_{,})}{(1 - n_{,})m}{\cdot}\frac{b}{br} + \frac{(n_{,} - i_{,})}{(1 - n_{,})}{\cdot}rc \quad . \quad . \quad (58)^*$$

$$r = \frac{n_{,}{\cdot}\theta_{,}(1 - \theta_{,}/2n_{,})}{(1 - n_{,})m} + \frac{(n_{,} - i_{,})}{(1 - n_{,})}{\cdot}rc \quad . \quad . \quad (58a)\dagger$$

$$rc = \frac{(1 - n_{,})\,m{\cdot}r - n_{,}{\cdot}\theta_{,}\,(1 - \theta_{,}/2n_{,})b/br}{(n_{,} - i_{,})m} \quad . \quad (59)^*$$

$$rc = \frac{(1 - n_{,})m{\cdot}r - n_{,}{\cdot}\theta_{,}(1 - \theta_{,}/2n_{,})}{(n_{,} - i_{,})m} \quad . \quad . \quad (59a)\dagger$$

$$n_{,} = \frac{\theta_{,}^{2}{\cdot}b/br + 2m(r + rc{\cdot}i_{,})}{2\theta_{,}{\cdot}b/br + 2m(r + rc)} \quad . \quad . \quad . \quad (60)^*$$

$$n_{,} = \frac{\theta_{,}^{2} + 2m(r + rc{\cdot}i_{,})}{2\theta_{,} + 2m(r + rc)} \quad . \quad . \quad . \quad . \quad (60a)\dagger$$

* For use in (48), (49), (50), (51), and (52).
† For use in (48*a*), (49*a*), (50*a*), (51*a*), and (52*a*).

ABBREVIATED FORMS OF EQUATIONS

All resistance moment formulæ containing the symbol n_1 can be written in an abbreviated form by proceeding as follows—

Let

a = arm of the resistance moment

a_1 = the arm ratio = a/d

Just as

n = neutral axis depth

n_1 = neutral axis depth ratio = n/d

Then, for rectangular beams and for tee beams of Class I

$$a = \left(d - \frac{n}{3}\right)$$

Dividing all terms on both sides by d, we get

$$\frac{a}{d} = \frac{d}{d} - \frac{1}{3} \cdot \frac{n}{d}$$

But

$$\frac{a}{d} = a_1 \text{ and } \frac{n}{d} = n_1$$

Therefore,

$$a_1 = \left(1 - \frac{n_1}{3}\right)$$

Consequently, formula (1)

$$Rt = \left[r\left(1 - \frac{n_1}{3}\right)t\right]b \cdot d^2$$

can be written in the form

$$Rt = [r \cdot a_1 \cdot t]b \cdot d^2$$

Other formulæ containing the expression $\left(1 - \frac{n_1}{3}\right)$ can be similarly abbreviated.

In cases where

$$a = d - \left(\frac{3n \cdot \theta - 2\theta^2}{6n - 3\theta}\right)$$

we can divide all terms on both sides by d and obtain

$$a_1 = \left(1 - \frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)$$

It follows that

$$1 - a_1 = \left(\frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right), \text{and} \left(\frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1} - i_1\right) = [1 - a_1 - i_1]$$

By re-arrangement, we have

$$1 - i_1 - a_1$$

and as

$$1 - i_1 = ac_1$$

we can write

$$(1 - i_1) - a_1 = ac_1 - a_1$$

Therefore formula (48*a*)

$$Rt = \left[r\left(1 - \frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)t + rc\left(\frac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1} - i_1\right)cs\right] b \cdot d^2$$

can be abbreviated thus

$$Rt = [r \cdot a_1 \cdot t + rc(ac_1 - a_1)cs]b \cdot d^2$$

Analagous equations can be similarly treated.

As abbreviated in the manner described, the various equations for resistance moments can frequently be used to save time and labour, and they can be written on the type-writer much more conveniently than in their original forms.

Simple Working Formulæ for all Classes of Beams.—On examination of the preceding formulæ for calculating the resistance moment of all classes of rectangular and T-beams, it will be noticed that symbols, such as b and d, denoting *breadth* and *depth* have been placed outside the square brackets comprising all other factors.

This system of arrangement has been adopted in the present work, partly with the objects of facilitating comparison of the formulæ for different classes of beams and of enabling the student or the user to realize the essential similarity of the equations, and partly for the purpose of providing for the establishment of simple working formulæ for everyday practice.

An exceedingly simple form of equation for the resistance moment is obtained if we express by the symbol Q all the *qualifiers* or qualifying factors other than $b{\cdot}d^2$, or $br{\cdot}d^2$, thus reducing formulæ such as

$$Rt = \left[r\left(1 - \frac{n_{\prime}}{3}\right)t\right]b{\cdot}d^2$$

to

$$Rt = Q{\cdot}b{\cdot}d^2$$

Similarly, the equation

$$Rc = \left[n_{\prime}\left(1 - \frac{n_{\prime}}{3}\right)\tfrac{1}{2}c\right]b{\cdot}d^2$$

can be written as

$$Rc = Q{\cdot}b{\cdot}d^2$$

and so on for all others of the series.

A simplified equation in the general form $R = Q{\cdot}b{\cdot}d^2$ can be employed with the aid of numerical coefficients calculated in advance for given working stresses in concrete and steel, or in conjunction with labour-saving diagrams in which are plotted values of Q and other qualifying factors.

Numerical Coefficients for Given Working Stresses.—Let us assume that the working stresses are to be 600 lb. per square inch for concrete and 16,000 lb. per square inch for steel, that $m = 15$, and that the economic ratio of reinforcement is to be adopted for beams of rectangular section.

Then we have

By (3) $$n_{\prime} = \frac{1}{1 + \dfrac{16000}{15 \times 600}} = 0{\cdot}36$$

By (6) $$r = \frac{{\cdot}36^2}{2 \times {\cdot}64 \times 15} = 0{\cdot}00675$$

By (1) $$R = \left[{\cdot}00675\left(1 - \frac{{\cdot}36}{3}\right)16000\right]b{\cdot}d^2 = 95b{\cdot}d^2$$

By (2) $$R = \left[{\cdot}36\left(1 - \frac{{\cdot}36}{3}\right)\tfrac{1}{2} \times 600\right]b{\cdot}d^2 = 95b{\cdot}d^2$$

For the working stresses and conditions stated, $r = 0{\cdot}00675$ and $Q = 95$ are constants which can be applied to the design of any number of beams of any dimensions. For other stresses and conditions similar constants can easily be calculated.

Labour-saving Diagrams.—A comparatively limited number of constants may serve the purposes of those who confine their operations to the design of beams in which only one or two sets of working stresses and few variations in the ratio of reinforcement are involved. But to cover the whole range of design, it is necessary to have at hand comprehensive tables, or diagrams from which any required constants can be taken.

Examples of Simple Working Formulæ.—By equating the formula $R = Q{\cdot}b{\cdot}d^2$ with any of the familiar bending moment equations, such as $B = \dfrac{W{\cdot}l}{4}$, $B = \dfrac{W{\cdot}l}{8}$ and so on, for different load conditions, working formulæ of particularly simple character can be derived for employment in conjunction with values of Q calculated in advance or taken from labour-saving tables or diagrams.

The subjoined rules for beams, with fixed ends and uniformly distributed loading, are given as examples of others which can readily be prepared for beams working under other conditions.

Equating the resistance and bending moments, we have

$$R = B = Q{\cdot}b{\cdot}d^2 = \frac{W{\cdot}l}{10}$$

Whence

$$W = \frac{10Q{\cdot}b{\cdot}d^2}{l} \qquad . \quad . \quad . \quad . \quad . \qquad (61)$$

$$l = \frac{10Q{\cdot}b{\cdot}d^2}{W} \qquad . \quad . \quad . \quad . \quad . \qquad (62)$$

$$b = \frac{W \cdot l}{10Q \cdot d^2} \qquad . \quad . \quad . \qquad (63)$$

$$d = \sqrt{\left(\frac{W \cdot l}{10Q \cdot b}\right)} \qquad . \quad . \quad . \quad . \quad . \qquad (64)$$

By adopting the proportions $b = \frac{2}{3}d$, or $d = 1\frac{1}{2}b$, the depth of a rectangular beam may be obtained directly from the following modification of formula (64)

$$d = \sqrt[3]{\left(\frac{1{\cdot}5W \cdot l}{10Q}\right)} \qquad . \quad . \quad . \quad . \qquad (64a)$$

This equation can be adapted for tee beams by including an expression for a predetermined value of the ratio b/br, and so becomes

$$d = \sqrt[3]{\left(\frac{1{\cdot}5W \cdot l}{10Q \cdot b/br}\right)} \qquad . \quad . \quad . \quad . \qquad (64b)$$

The values of any constants, in addition to that of Q, which may be required for the purposes of design have necessarily to be calculated before arriving at the values of Q, and should be included in the tables prepared for reference.

SUMMARY OF RESISTANCE MOMENT FORMULÆ

WITH NUMERICAL EXAMPLES

In the following pages, the various series of formulæ stated in this chapter for the resistance moments of beams are summarized in tabular form, and numerical examples of the equations are given on the pages facing the respective tables.

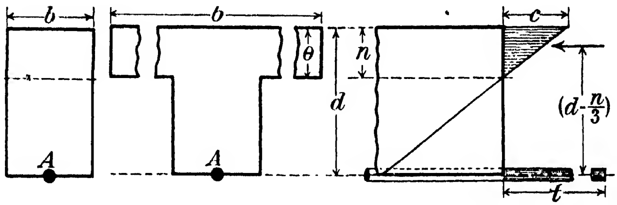
The data upon which the examples have been calculated have been taken on a uniform basis, as far as possible, in order that the results may be compared, and that conclusions may be drawn as to the relative merits of different classes of beam design.

A table on page 158 contains a summary of the calculated results for all types of beams for which formulæ are given, and shows the values of the resistance moments in terms of the areas of concrete and steel required, as well as the relative economy of each type of design.

SUMMARY OF BEAM FORMULÆ

RECTANGULAR BEAMS AND T-BEAMS (CLASS I)

SINGLE REINFORCEMENT



Rectangular and T-Beams ($r = A/b{\cdot}d$).	Formula No.
$Rt = [r(1 - \frac{1}{3}n_i)t]b{\cdot}d^2$	(1) (8a)
$Rc = [n_i(1 - \frac{1}{3}n_i)\frac{1}{2}c]b{\cdot}d^2$	(2) (9)
$n_i = \dfrac{1}{1 + (t/m{\cdot}c)}$	(3) (10)
$t = \dfrac{(1 - n_i)}{n_i}{\cdot}m{\cdot}c$	(4) (11)
$c = \dfrac{n_i}{(1 - n_i)}{\cdot}\dfrac{t}{m}$	(5) (12)
$r = \dfrac{n_i^2}{2(1 - n_i)m}$	(6) (13a)
$n_i = \sqrt{([m{\cdot}r]^2 + 2m{\cdot}r)} - m{\cdot}r$	(7) (14)
T-Beams only ($r = A/br{\cdot}d$).	
$Rt = [r(1 - \frac{1}{3}n_i)t]br{\cdot}d^2$	(8)
$Rc = [n_i(1 - \frac{1}{3}n_i)\frac{1}{2}c]b{\cdot}d^2$	(9)
$r = \dfrac{n_i^2}{2(1 - n_i)m}{\cdot}\dfrac{b}{br}$	(13)
$n_i = \sqrt{([m{\cdot}r{\cdot}br/b]^2 + [2m{\cdot}r{\cdot}br/b])} - [m{\cdot}r{\cdot}br/b]$	(14)

NUMERICAL EXAMPLES

RECTANGULAR BEAMS AND T-BEAMS (CLASS I)

SINGLE REINFORCEMENT

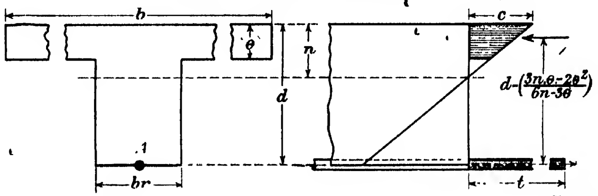
DATA.—$c = 600$ lb./in.^{2}; $t = 16000$ lb./in.^{2}; $b = 10$ in. (Rectangular beam); $b = 40$ in. (Tee beam); $br = 10$ in. (Tee beam); $d = 15$ in. (Rectangular and Tee beams).

Rectangular and T-Beams ($r = A/b{\cdot}d$).	Formula No.
$n_1 = \dfrac{1}{1 + 16000/15 \times 600} = 0{\cdot}36$ . . .	(3) (10)
$t = \dfrac{(1 - {\cdot}36)}{{\cdot}36} \times 15 \times 600 = 16000$ (lb./in.^{2}). .	(4) (11)
$c = \dfrac{{\cdot}36}{1 - {\cdot}36} \times \dfrac{16000}{15} = 600$ (lb./in.^{2}) . .	(5) (12)
$r = \dfrac{{\cdot}36^2}{2(1 - {\cdot}36)15} = 0{\cdot}00675$ . . .	(6) (13a)
$n_1 = \sqrt{([15 \times {\cdot}00675]^2 + 2 \times 15 \times {\cdot}00675)} - [15 \times {\cdot}00675] = 0{\cdot}36$ . . .	(7) (14a)
$Rt = [{\cdot}00675(1 - {\cdot}12)16000]b{\cdot}d^2$	(1)
$= 95b{\cdot}d^2$	(8a)
$Rc = [{\cdot}36(1 - {\cdot}12)300]b{\cdot}d^2$	(2)
$= 95b{\cdot}d^2 = 213{,}750$ in.lb.(Rect.) $= 855{,}000$ in.lb.(T)	(9)
T-Beams only ($r = A/br{\cdot}d$).	
$r = \dfrac{{\cdot}36^2}{2(1 - {\cdot}36)15} \times 4 = 0{\cdot}027$	(13)
$n_1 = \sqrt{[15 \times {\cdot}027 \times \tfrac{1}{4}]^2 + [2 \times 15 \times {\cdot}027 \times \tfrac{1}{4}]} - [15 \times {\cdot}027 \times \tfrac{1}{4}] = 0{\cdot}36$	(14)
$Rt. = [{\cdot}027(1 - {\cdot}12)16000]\,br{\cdot}d^2 = 380\,br{\cdot}d^2 = 855{,}000$ in.-lb.	(8)
$Rc = [{\cdot}36(1 - {\cdot}12)300]b{\cdot}d^2 = 95b{\cdot}d^2 = 855{,}000$ in.-lb.	(9)

SUMMARY OF BEAM FORMULÆ (*continued*)

T-BEAMS (CLASS II)

SINGLE REINFORCEMENT



Ratio of Reinforcement: $r = A/b{\cdot}d$. | Formula No.

$$Rt = \left[r\left(1 - \frac{3n_{\prime}\cdot\theta_{\prime} - 2\theta_{\prime}^2}{6n_{\prime} - 3\theta_{\prime}}\right)t\right]b{\cdot}d^2 \qquad (15a)$$

$$Rc = \left[\theta_{\prime}\left(1 - \frac{3n_{\prime}\cdot\theta_{\prime} - 2\theta_{\prime}^2}{6n_{\prime} - 3\theta_{\prime}}\right)\left(1 - \frac{\theta_{\prime}}{2n_{\prime}}\right)c\right] b{\cdot}d^2 \qquad (16)$$

$$n_{\prime} = \frac{1}{1 + t/m{\cdot}c} \qquad (17)$$

$$t = \frac{(1 - n_{\prime})}{n_{\prime}}{\cdot}m{\cdot}c \qquad (18)$$

$$c = \frac{n_{\prime}}{(1 - n_{\prime})}\cdot\frac{t}{m} \qquad (19)$$

$$r = \frac{n_{\prime}\cdot\theta_{\prime}(1 - \theta_{\prime}/2n_{\prime})}{(1 - n_{\prime})m} \qquad (20a)$$

$$n_{\prime} = \frac{2m{\cdot}r + \theta_{\prime}^2}{2m{\cdot}r + 2\theta_{\prime}} \qquad (21a)$$

Ratio of Reinforcement: $r = A/br{\cdot}d$.

$$Rt = \left[r\left(1 - \frac{3n_{\prime}\cdot\theta_{\prime} - 2\theta_{\prime}^2}{6n_{\prime} - 3\theta_{\prime}}\right)t\right] br{\cdot}d^2 \qquad (15)$$

$$Rc = \left[\theta_{\prime}\left(1 - \frac{3n_{\prime}\cdot\theta_{\prime} - 2\theta_{\prime}^2}{6n_{\prime} - 3n_{\prime}}\right)\left(1 - \frac{\theta_{\prime}}{2n_{\prime}}\right) c{\cdot}b/br\right] br{\cdot}d^2 \qquad (16)$$

$$r = \frac{n_{\prime}\cdot\theta_{\prime}(1-\theta_{\prime}/2n_{\prime})}{(1 - n_{\prime})m}\cdot\frac{b}{br} \qquad (20)$$

$$n_{\prime} = \frac{2m{\cdot}r + \theta_{\prime}^2{\cdot}(b/br)}{2m{\cdot}r + 2\theta_{\prime}(b/br)} \qquad (21)$$

NUMERICAL EXAMPLES

T-BEAMS (CLASS II)

SINGLE REINFORCEMENT

DATA.—$c = 600$ lb./in.^{2}; $t = 16000$ lb./in.^{2}; $b = 40$ in.; $br = 10$ in.; $d = 15$ in.; $\theta_i = 0{\cdot}3$.

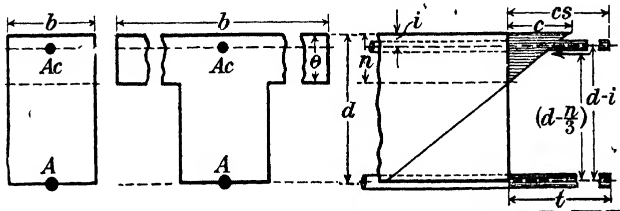
Ratio of Reinforcement: $r = A/b{\cdot}d$.	Formula No.
$n_i = 0{\cdot}36$ as before	(17)
$t = 16000$ (lb./in.^{2}) as before	(18)
$c = 600$ (lb./in.^{2}) as before.	(19)
$r = \dfrac{{\cdot}36 \times {\cdot}3(1 - {\cdot}3/{\cdot}72)}{(1 - {\cdot}36)15} = 0{\cdot}00656$ . . .	(20a)
$n_i = \dfrac{2\times15\times{\cdot}00656+{\cdot}3^2}{2\times15\times{\cdot}00656+2\times{\cdot}3} = 0{\cdot}36$	(21a)
$Rt = \left[{\cdot}00656\left(1 - \dfrac{3\times{\cdot}36\times{\cdot}3 - 2\times{\cdot}3^2}{6\times{\cdot}36 - 3\times{\cdot}3}\right)16000\right]b{\cdot}d^2$	
$= 93b{\cdot}d^2 = 837{,}000$ in.-lb.	(15a)
$Rc = [{\cdot}3(1 - {\cdot}1143)(1 - {\cdot}3/{\cdot}72)\ 600]b{\cdot}d^2$	
$= 93b{\cdot}d^2 = 837{,}000$ in.-lb.	(16)

Ratio of Reinforcement: $r = A/br{\cdot}d$.	
$r = \dfrac{{\cdot}36 \times {\cdot}3(1 - {\cdot}3/{\cdot}72)4}{(1 - {\cdot}36)15} = 0{\cdot}02624$ . . .	(20)
$n_i = \dfrac{2 \times 15 \times{\cdot}02624 + {\cdot}3^2 \times 4}{2 \times 15 \times{\cdot}02624 + 2 \times {\cdot}3 \times 4} = 0{\cdot}36$. . .	(21)
$Rt = [{\cdot}02624\ (1 - {\cdot}1143)\ 16000]br{\cdot}d^2$	
$= 372br{\cdot}d^2 = 837{,}000$ in.-lb.	(15)
$Rc = [{\cdot}3(1 - {\cdot}1143)(1 - {\cdot}3/{\cdot}72)\ 600 \times 4]\ br{\cdot}d^2$	
$= 372br{\cdot}d^2 = 837{,}000$ in.-lb.	(16)

SUMMARY OF BEAM FORMULÆ (*continued*)

RECTANGULAR BEAMS AND T-BEAMS (CLASS I)

DOUBLE REINFORCEMENT



Ratio of Reinforcement : $r = A/b \cdot d$.

	Formula No.
$Rt = [r(1 - \frac{1}{3}n_{,})t + rc(\frac{1}{3}n_{,} - i_{,})cs]b \cdot d^2$	(22) (35a)
$Rt = [r(1 - i_{,})t - n_{,}(\frac{1}{3}n_{,} - i_{,})\frac{1}{2}c]b \cdot d^2$	(23) (36a)
$Rc = [n_{,}(1 - \frac{1}{3}n_{,})\frac{1}{2}c + rc(1 - i_{,})cs]b \cdot d^2$	(24) (37a)
For Beams where $i_{,} = \frac{1}{3}n_{,}$	
$Rt = [r(1 - \frac{1}{3}n_{,})t]b \cdot d^2$	(25) (38a)
$Rc = [(n_{,} \cdot \frac{1}{2}c + rc \cdot cs)(1 - \frac{1}{3}n_{,})]b \cdot d^2$	(26) (39a)
Formulæ (27) to (29) and (40) to (42) are the same as (3) to (5)	
$cs = \frac{(n_{,} - i_{,})}{n_{,}} \cdot m \cdot c$	(30) (43)
$cs = \frac{(n_{,} - i_{,})}{(1 - n_{,})} \cdot t$	(31) (44)
$r = \frac{n_{,}^2}{2(1 - n_{,})m} + \frac{(n_{,} - i_{,})}{(1 - n_{,})} \cdot rc$	(32) (45a)
$rc = \frac{2(1 - n_{,})m \cdot r - n_{,}^2}{2(n_{,} - i_{,})m}$	(33) (46a)
$n_{,} = \sqrt{([m(r + rc)]^2 + 2m[r + rc \cdot i_{,}])} - m[r + rc]$	(34) (47a)

NUMERICAL EXAMPLES

RECTANGULAR BEAMS AND T-BEAMS (CLASS 1)

DOUBLE REINFORCEMENT

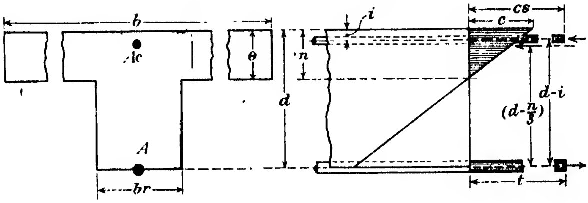
DATA.—$c = 600$ lb./in.^{2}; $t = 16000$ lb./in.^{2}; $rc = 0{\cdot}00675$; $n_1 = 0{\cdot}36$; $b = 10$ in. (Rect.), $b = 40$ in. (Tee), $d = 15$ in. (Rect.) and (Tee).

Ratio of Reinforcement: $r = A/b{\cdot}d$; $i_1 = 0{\cdot}1$.	Formula No.
$cs = \left(\dfrac{{\cdot}36 - {\cdot}1}{{\cdot}36}\right) \times 9000 - \dfrac{{\cdot}26}{{\cdot}64} \times 16000 = 6500 (\text{lb./in}^2)$	(30) (31)
$r = \dfrac{{\cdot}36^2}{2(1 - {\cdot}36)15} + \dfrac{({\cdot}36 - {\cdot}1)}{(1 - {\cdot}36)} \times {\cdot}00675 = 0{\cdot}00949$	(32) (45a)
$rc = \dfrac{2(1 - {\cdot}36)15 \times {\cdot}00949 - {\cdot}36^2}{2 \times ({\cdot}36 - {\cdot}1)15} = 0{\cdot}00675$ . .	(33) (46a)
$n_1 = \sqrt{([15 \times {\cdot}01624]^2 + 30 \times {\cdot}01016)} - 15 \times {\cdot}01624 = 0{\cdot}36$	(34) (37a)
$Rt = [{\cdot}00949(1 - {\cdot}12)16000 + {\cdot}00675({\cdot}12 - {\cdot}1)6500]b{\cdot}d^2$	(22)
$= 302{,}625$ in.-lb. (Rect.) $= 1{,}210{,}500$ in.-lb. (Tee)	(35a)
$Rt = [{\cdot}00949(1 - {\cdot}1)16000 - {\cdot}36({\cdot}12 - {\cdot}1)300]b{\cdot}d^2$	(23)
$= 302{,}625$ in.-lb. (Rect.) $= 1{,}210{,}500$ in.-lb. (Tee)	(36a)
$Rc = [{\cdot}36(1 - {\cdot}12)300 + {\cdot}00675(1 - {\cdot}1)6500]b{\cdot}d^2$	(24)
$= 302{,}625$ in.-lb. (Rect.) $= 1{,}210{,}500$ in.-lb. (Tee)	(37a)
For Beams where $i_1 = \frac{1}{3}n_1$	
$cs = \left(\dfrac{{\cdot}36 - {\cdot}12}{{\cdot}36}\right) \times 9000 - \dfrac{{\cdot}24}{{\cdot}64} \times 16000 = 6000 (\text{lb./in.}^2)$	(30) (31)
$r = \dfrac{{\cdot}36^2}{2(1 - {\cdot}36)15} + \dfrac{({\cdot}36 - {\cdot}12)}{(1 - {\cdot}36} \times {\cdot}00675 = 0{\cdot}00928$	(32) (45a)
$Rt = [{\cdot}00928(1 - {\cdot}12)16000]b{\cdot}d^2$	(25)
$= 293{,}625$ in.-lb. (Rect.) $= 1{,}174{,}500$ in.-lb. (Tee)	(38a)
$Rc = [({\cdot}36 \times 300 + {\cdot}00675 \times 6000)(1 - {\cdot}12)]b{\cdot}d^2$	(26)
$= 293{,}625$ in.-lb. (Rect.) $= 1{,}174{,}500$ in.-lb. (Tee)	(39a)

SUMMARY OF BEAM FORMULÆ (*continued*)

T-BEAMS (CLASS I)

DOUBLE REINFORCEMENT



Ratio of Reinforcement : $r = A/br\cdot d$.

	Formula No.
$Rt = [r(1 - \frac{1}{3}n_1)t + rc(\frac{1}{3}n_1 - i_1)cs]br\cdot d^2$	(35)
$Rt = [r(1 - i_1)t - n_1(\frac{1}{3}n_1 - i_1)\frac{1}{2}c\cdot b/br]br\cdot d^2$	(36)
$Rc = [n_1(1 - \frac{1}{3}n_1)\frac{1}{2}c\cdot b/br + rc(1 - i_1)cs]br\cdot d^2$	(37)
For Beams where $i_1 = \frac{1}{3}n_1$	
$Rt = [r(1 - \frac{1}{3}n_1)t]br\cdot d^2$	(38)
$Rc = [(n_1\cdot\frac{1}{2}c\cdot b/br + rc\cdot cs)(1 - \frac{1}{3}n_1)]br\cdot d^2$	(39)

Formulæ (40) to (42) are the same as (3) to (5).

$cs = \dfrac{(n_1 - i_1)}{n_1}\cdot m\cdot c$	(43)
$cs = \dfrac{(n_1 - i_1)}{(1 - n_1)}\cdot t$	(44)
$r = \dfrac{n_1^2}{2(1 - n_1)m}\cdot\dfrac{b}{br} + \dfrac{(n_1 - i_1)}{(1 - n_1)}\cdot rc$	(45)
$rc = \dfrac{2(1 - n_1)m\cdot r - n_1^2\cdot b/br}{2(n_1 - i_1)m}$	(46)
$n_1 = \sqrt{([m(r+rc)br/b]^2 + [2m(r + rc\cdot i_1)br/b])} - [m(r + rc)br/b]$	(47)

NUMERICAL EXAMPLES

T-BEAMS (CLASS I)

DOUBLE REINFORCEMENT

DATA.—$c = 600$ lb./in.^{2}; $t = 16000$ lb./in.^{2}; $rc = 0{\cdot}027$; $n_t = 0{\cdot}36$; $b = 40$ in.; $br = 10$ in., $d = 15$ in.

Ratio of Reinforcement: $r = A/br{\cdot}d$; $i_t = 0{\cdot}1$. | Formula No.

$$cs = \frac{({\cdot}36 - {\cdot}1)}{{\cdot}36} \times 9000 = \frac{{\cdot}26}{{\cdot}64} \times 16000 = 6500 (\text{lb./in}^2) \qquad (43)\ (44)$$

$$r = \frac{{\cdot}36^2 \times 4}{2(1 - {\cdot}36)15} + \frac{({\cdot}36 - {\cdot}1)}{(1 - {\cdot}36)} \times {\cdot}027 = 0{\cdot}03796 \qquad (45)$$

$$rc = \frac{2(1 - {\cdot}36)15 \times {\cdot}03796 - {\cdot}36^2 \times 4}{2({\cdot}36 - {\cdot}1)15} = 0{\cdot}027 \qquad (46)$$

$$n_t = \sqrt{\left[\left(\frac{15 \times {\cdot}065}{4}\right)^2 + \left(\frac{30 \times {\cdot}0406}{4}\right)\right]} - \left(\frac{15 \times {\cdot}065}{4}\right) \qquad (47)$$

$$= 0{\cdot}36$$

$$Rt = [{\cdot}03796(1 - {\cdot}12)16000 + {\cdot}027({\cdot}12 - {\cdot}1)6500]br{\cdot}d^2$$
$$= 538br{\cdot}d^2 = 1{,}210{,}500 \text{ in.-lb.} \qquad (35)$$

$$Rt = [{\cdot}03796(1 - {\cdot}1)16000 - {\cdot}36({\cdot}12 - {\cdot}1)300 \times 4]br{\cdot}d^2$$
$$= 538br{\cdot}d^2 = 1{,}210{,}500 \text{ in.-lb.} \qquad (36)$$

$$Rc = [{\cdot}36(1 - {\cdot}12)300 \times 4 + {\cdot}027(1 - {\cdot}1)6500]br{\cdot}d^2$$
$$= 538br{\cdot}d^2 = 1{,}210{,}500 \text{ in.-lb.} \qquad (37)$$

For Beams where $i_t = \frac{1}{3}n_t$

$$cs = \frac{({\cdot}36 - {\cdot}12)}{{\cdot}36} \times 9000 = \frac{{\cdot}24}{{\cdot}64} \times 16000 = 6000 (\text{lb./in.}^2) \qquad (43)\ (44)$$

$$r = \frac{{\cdot}36^2 \times 4}{2(1 - {\cdot}36)15} + \frac{({\cdot}36 - {\cdot}12)}{(1 - {\cdot}36)} \times {\cdot}027 = 0{\cdot}03712 \qquad (45)$$

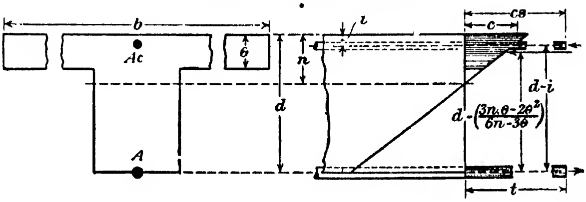
$$Rt = [{\cdot}03712(1 - {\cdot}12)16000]br{\cdot}d^2$$
$$= 522br{\cdot}d^2 = 1{,}174{,}500 \text{ in.-lb.} \qquad (38)$$

$$Rc = [({\cdot}36 \times 300 \times 4 + {\cdot}027 \times 6000)(1 - {\cdot}12)]br{\cdot}d^2$$
$$= 522br{\cdot}d^2 = 1{,}174{,}500 \text{ in.-lb.} \qquad (39)$$

SUMMARY OF BEAM FORMULÆ (*continued*)

T-BEAMS (CLASS II)

DOUBLE REINFORCEMENT



Ratio of Reinforcement : $r = A/b{\cdot}d$.

Formula No.

$$Rt = \left[r\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^2}{6n_{,} - 3\theta_{,}}\right)t + rc\left(\frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^2}{6n_{,} - 3\theta_{,}} - i_{,}\right)cs\right]b{\cdot}d^2 \qquad (48a)$$

$$Rt = \left[r\left(1 - i_{,}\right)t - \theta_{,}\left(\frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^2}{6n_{,} - 3\theta_{,}} - i_{,}\right)\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c\right]b{\cdot}d^2 \qquad (49a)$$

$$Rc = \left[\theta_{,}\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^2}{6n_{,} - 3\theta_{,}}\right)\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c + rc\left(1 - i_{,}\right)cs\right]b{\cdot}d^2 \qquad (50a)$$

For Beams where $i_{,} = (3n_{,}\cdot\theta_{,} - 2\theta_{,}^2)/(6n_{,} - 3\theta_{,})$

$$Rt = \left[r\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^2}{6n_{,} - 3\theta_{,}}\right)t\right]b{\cdot}d^2 \quad . \quad . \quad . \quad . \qquad (51a)$$

$$Rc = \left[\left\{\theta_{,}\left(1 - \frac{\theta_{,}}{2n_{,}}\right)c + rc{\cdot}cs\right\}\left(1 - \frac{3n_{,}\cdot\theta_{,} - 2\theta_{,}^2}{6n_{,} - 3\theta_{,}}\right)\right]b{\cdot}d^2 \,. \qquad (52a)$$

Formulæ (53) to (55) are the same as (3) to (5)

$$cs = \frac{(n_{,} - i_{,})}{n_{,}}\cdot m{\cdot}c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

$$cs = \frac{(n_{,} - i_{,})}{(1 - n_{,})}\cdot t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

$$r = \frac{n_{,}\cdot\theta_{,}(1 - \theta_{,}/2n_{,})}{(1 - n_{,})m} + \frac{(n_{,} - i_{,})}{(1 - n_{,})}\cdot rc \quad . \quad . \quad . \quad . \qquad (58a)$$

$$rc = \frac{(1 - n_{,})\, m{\cdot}r - n_{,}\cdot\theta_{,}(1 - \theta_{,}/2n_{,})}{(n_{,} - i_{,})m} \quad . \quad . \quad . \quad . \qquad (59a)$$

$$n_{,} = \frac{\theta_{,}^2 + 2m(r + rc{\cdot}i_{,})}{2\theta_{,} + 2m(r + rc)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (60a)$$

NUMERICAL EXAMPLES

T-BEAMS (CLASS II)

DOUBLE REINFORCEMENT

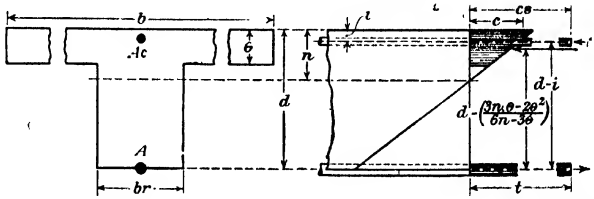
DATA.—$c = 600$ lb./in.^{2}; $t = 16000$ lb./in.^{2}; $\theta_{\prime} = 0{\cdot}3$; $rc = {\cdot}00675$; $n_{\prime} = 0{\cdot}36$, as previously calculated; $b = 40$in.; $d = 15$ in.

Ratio of Reinforcement: $r = A/b{\cdot}d$; $i_{\prime} = 0{\cdot}1$.	Formula No.
$cs = \dfrac{({\cdot}36-{\cdot}1)}{{\cdot}36}\times 9000 = \dfrac{({\cdot}26)}{({\cdot}64)}\times 16000 = 6500$ (lb./in.^{2})	(56) (57)
$r = \dfrac{{\cdot}36\times{\cdot}3(1-{\cdot}3/{\cdot}72)}{(1-{\cdot}36)15}+\dfrac{({\cdot}36-{\cdot}1)}{(1-{\cdot}36)}\times{\cdot}00675 = 0{\cdot}0093$	(58a)
$rc = \dfrac{(1-{\cdot}36)15\times{\cdot}0093-{\cdot}36\times{\cdot}3(1-{\cdot}3/{\cdot}72)}{({\cdot}36-{\cdot}1)15} = 0{\cdot}00675$	(59a)
$n_{\prime} = \dfrac{{\cdot}3^2+2\times15({\cdot}0093+{\cdot}00675\times{\cdot}1)}{2\times{\cdot}3+2\times15({\cdot}0093+{\cdot}00675)} = 0{\cdot}36$ . .	(60a)
$Rt = [{\cdot}0093\times{\cdot}8857\times16000+{\cdot}00675\times{\cdot}0143 \times 6500]b{\cdot}d^2 = 132{\cdot}5b{\cdot}d^2 = 1{,}192{,}500$ in.-lb.	(48a)
$Rt = [{\cdot}0093\times{\cdot}9\times16000-{\cdot}3\times{\cdot}0143\times{\cdot}58\times600]b{\cdot}d^2 = 132{\cdot}5b{\cdot}d^2 = 1{,}192{,}500$ in.-lb.	(49a)
$Rc = [{\cdot}3\times{\cdot}8857\times{\cdot}58\times600+{\cdot}00675\times{\cdot}9\times6500]b{\cdot}d^2 = 132{\cdot}5b{\cdot}d^2 = 1{,}192{,}500$ in.-lb.	(50a)
For Beams where $i_{\prime} = (3n_{\prime}{\cdot}\theta_{\prime} - 2\theta_{\prime}^2)/(6n_{\prime} - 3\theta_{\prime})$	
$cs = \dfrac{({\cdot}36-{\cdot}114)}{{\cdot}36}\times9000 = \dfrac{{\cdot}246}{{\cdot}64}\times16000 = 6140$(lb./in.^{2})	(56) (57)
$r = \dfrac{{\cdot}36\times{\cdot}3\times{\cdot}58}{(1-{\cdot}36)15}+\dfrac{{\cdot}246}{{\cdot}64}\times{\cdot}00675 = 0{\cdot}00915$	(58a)
$Rt = [{\cdot}00915\times{\cdot}8857\times16000]b{\cdot}d^2 = 129{\cdot}5b{\cdot}d^2 = 1{,}165{,}500$ in.-lb.	(51a)
$Rc = [\{{\cdot}3\times{\cdot}58\times600+{\cdot}00675\times6140\}\times{\cdot}8857]b{\cdot}d^2 = 129{\cdot}5b{\cdot}d^2 = 1{,}165{,}500$ in.-lb.	(52a)

SUMMARY OF BEAM FORMULÆ (*continued*)

T-BEAMS (CLASS II)

DOUBLE REINFORCEMENT



Ratio of Reinforcement: $r = A/br \cdot d$.

	Formula No.
$Rt = \left[r\left(1 - \dfrac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)t + rc\left(\dfrac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1} - i_1\right)cs\right]br \cdot d^2$	(48)
$Rt = \left[r\left(1 - i_1\right)t - \theta_1\left(\dfrac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1} - i_1\right)\left(1 - \dfrac{\theta_1}{2n_1}\right)c \cdot b/br\right]br \cdot d^2$	(49)
$Rc = \left[\theta_1\left(1 - \dfrac{3n_1 \cdot \theta - 2\theta_1^2}{6n_1 - 3\theta_1}\right)\left(1 - \dfrac{\theta_1}{2n_1}\right)c \cdot b/br + rc\left(1 - i_1\right)cs\right]br \cdot d^2$	(50)
For Beams where $i_1 = (3n_1 \cdot \theta_1 - 2\theta_1^2)/(6n_1 - 3\theta_1)$	
$Rt = \left[r\left(1 - \dfrac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)t\right]br \cdot d^2$	(51)
$Rc = \left[\left\{\theta_1\left(1 - \dfrac{\theta_1}{2n_1}\right)c \cdot b/br + rc \cdot cs\right\}\left(1 - \dfrac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)\right]br \cdot d^2$	(52)
Formulæ (53) to (55) are the same as (3) to (5)	
$cs = \dfrac{(n_1 - i_1)}{n_1} \cdot m \cdot c = cs = \dfrac{(n_1 - i_1)}{(1 - n_1)} \cdot t$	(56) (57)
$r = \dfrac{n_1 \cdot \theta_1(1 - \theta_1/2n_1)}{(1 - n_1)m} \cdot \dfrac{b}{br} + \dfrac{(n_1 - i_1)}{(1 - n_1)} \cdot rc$	(58)
$rc = \dfrac{(1 - n_1)m \cdot r - n_1 \cdot \theta_1(1 - \theta_1/2n_1)b/br}{(n_1 - i_1)m}$	(59)
$n_1 = \dfrac{\theta_1^2 \cdot b/br + 2m(r + rc \cdot i_1)}{2\theta_1 \cdot b/br + 2m(r + rc)}$	(60)

NUMERICAL EXAMPLES

T-BEAMS (CLASS II)

DOUBLE REINFORCEMENT

DATA.—$c = 600$ lb./in²; $t = 16000$ lb./in²; $b/br = 4$; $\theta_1 = 0{\cdot}3$; $rc = 0{\cdot}027$; $n_1 = 0{\cdot}36$; $b = 40$ in.; $br = 10$ in.; $d = 15$ in.

Ratio of Reinforcement, $r = A/br{\cdot}d$, $i_1 = 0{\cdot}1$ — Formula No.

$cs = 6500$ lb./in.², as previously calculated . . (56) (57)

$$r = \frac{{\cdot}36 \times {\cdot}3(1 - {\cdot}3/{\cdot}72)4}{(1 - {\cdot}36)15} + \frac{({\cdot}36 - {\cdot}1) \times {\cdot}027}{(1 - {\cdot}36)} = 0{\cdot}0372 \quad . \qquad (58)$$

$$rc = \frac{({\cdot}64 \times 15 \times {\cdot}0372 - {\cdot}36 \times {\cdot}3 \times {\cdot}58 \times 4}{({\cdot}36 - {\cdot}1)15} = 0{\cdot}027 \quad . \qquad (59)$$

$$n_1 = \frac{{\cdot}3 \times 4 + 2 \times 15({\cdot}0372 + {\cdot}027 \times {\cdot}1)}{2 \times {\cdot}3 \times 4 + 2 \times 15({\cdot}0372 + {\cdot}027)} = 0{\cdot}36 \quad . \qquad (60)$$

$$Rt = [{\cdot}0372 \times {\cdot}8857 \times 16000 + {\cdot}027 \times {\cdot}0143 \times 6500] \times br{\cdot}d^2 = 530\, br{\cdot}d^2 = 1{,}192{,}500 \text{ in.-lb.} \qquad (48)$$

$$Rt = [{\cdot}0372 \times {\cdot}9 \times 16000 - {\cdot}3 \times {\cdot}0143 \times {\cdot}58 \times 600 \times 4]\, br{\cdot}d^2 = 530 br{\cdot}d^2 = 1{,}192{,}500 \text{ in.-lb.} \qquad (49)$$

$$Rc = [{\cdot}3 \times {\cdot}8857 \times {\cdot}58 \times 600 \times 4 + {\cdot}027(1 - {\cdot}1) \times 6500]\, br{\cdot}d^2 = 530\, br{\cdot}d^2 = 1{,}192{,}500 \text{ in.-lb.} \qquad (50)$$

For Beams where $i_1 = (3n_1{\cdot}\theta_1 - 2\theta_1^2)/(6n_1 - 3\theta_1)$

$cs = 6140$ lb./in.², as previously calculated . . (56) (57)

$$r = \frac{{\cdot}36 \times {\cdot}3 \times {\cdot}58 \times 4}{(1 - {\cdot}36)15} + \left(\frac{{\cdot}36 - {\cdot}114}{1 - {\cdot}36}\right) \times {\cdot}027 = 0{\cdot}0366 \quad . \qquad (58)$$

$$Rt = [{\cdot}0366(1 - {\cdot}1143)16000]br{\cdot}d^2 = 518\, br{\cdot}d^2 = 1{,}165{,}500 \text{ in.-lb.} \qquad (51)$$

$$Rc = [\{{\cdot}3 \times 58 \times 600 \times 4 + {\cdot}027 \times 6140\}(1 - {\cdot}1143)] \times br{\cdot}d^2 = 518\, br{\cdot}d^2 = 1{,}165{,}500 \text{ in.-lb.} \qquad (52)$$

COMPARISON OF RESISTANCE MOMENTS FOR RECTANGULAR AND TEE BEAMS OF DIFFERENT TYPES, WITH SINGLE AND DOUBLE REINFORCEMENT

R = Resistance moment; AC = Net area of concrete; AS = total area of steel. The relative cost in Column (5) is based upon the assumption that the cost of steel is 50 times that of concrete for equal volumes.

Type of Beam.	Resistance Moment R	Concrete Total area AC	Steel Total area AS	Relative Cost AC+50AS	$\frac{R}{AC}$	$\frac{R}{AS}$	$\frac{R}{AC+50AS}$	Page Ref.
SINGLE REINFT.	in.-lb.	sq. in.	sq. in.		in.-lb.	in.-lb.	in.-lb.	
Rectangular .	213,750	148·98	1·0125	199·605	1435	211,100	1071	147
Tee (Class I) .	855,000	307·95	4·05	510·45	2776	211,100	1675	147
Tee (Class II) .	837,000	281·06	3·94	478·06	2978	212,400	1751	149
DOUBLE REINFT.								
Rectangular .	302,625	147·56	2·44	269·56	2051	124,000	1123	151
Tee (Class I) .	1,210,500	302·24	9·76	790·24	4005	124,000	1532	151/3
Tee (Class II) .	1,192,500	275·37	9·63	756·87	4331	123,830	1575	155/7
(WHERE $ac = a$) .								
Rectangular .	293,625	147·6	2·4	267·6	1989	122,300	1097	151
Tee (Class I) .	1,174,500	302·4	9·6	782·4	3884	122,300	1501	151/3
Tee (Class II) .	1,165,500	275·46	9·54	752·46	4232	122,160	1549	155/7
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

COMPARISON OF RESISTANCE MOMENTS

CALCULATED IN THE PRECEDING EXAMPLES

The resistance moments calculated for beams of different types are given in Col. (2) on the opposite page. The net area of concrete (AC) for each beam, deducting the amount displaced by the area of steel (AS), is given in Col. (3), and the total area of steel for each beam is given in Col. (4). The "relative cost" in Col. (5) is based upon the usual assumption that the cost of steel is 50 times the cost of concrete for equal volumes. Therefore for a beam where the net area of concrete AC = 99 and the total area of steel AS = 1, we should have AC + 50 AS = 149 as a basis of comparison. In Cols. (6), (7), and (8) resistance moments are given per square inch of concrete, per square inch of steel, and per unit of "relative cost."

The saving of concrete effected by the adoption of a tee-shaped cross section, particularly in conjunction with double reinforcement, is clearly shown by the figures in Col. (6).

So far as concerns economy in the use of steel, the results in Col. (7) are conclusive as to the high efficiency of single reinforcement in itself, but this advantage is counterbalanced in varying measure by the fact that the area of concrete required in singly reinforced beams is greater than that necessary in beams having double reinforcement.

Nevertheless, as shown in Col. (8), the two most economical types of beams are those of tee shape with single reinforcement. The list is headed by the tee beam of Class II, the kindred beam of Class I coming next in order of economy.

As will be seen on reference to the numerical examples, the higher efficiency of the Class II beam, as compared with that of Class I, is due to the greater length of the arm of resistance moments and the higher mean compressive stress in the concrete. Thus, in one case we have $a = 0{\cdot}8857 \times 15 = 13{\cdot}285$ (inches) and $cm = 350$ (lb./in.^{2}), and in the other case $a = 0{\cdot}88 \times 15 = 13{\cdot}2$ (inches) and $cm = 300$ (lb./in.^{2}).

The other figures in Col. (8) make clear the points, (1) that compression reinforcement can be employed in tee beams

with results nearly as favourable as those given by tension reinforcement alone, and (9) that doubly reinforced rectangular beams are more advantageous than similar beams with single reinforcement.

It should be noted that the types of beams in the three last lines of the table are less efficient than the corresponding types in the preceding three lines, the lower results being due to the shorter arm of resistance moments. In the beams on lines (4), (5), and (6) the compression reinforcement is placed so that $ac = 0{\cdot}9 \times 15 = 13{\cdot}5$ (inches), and in those on lines (7), (8), and (9) the compression reinforcement is disposed so that we have $ac = 0{\cdot}88 \times 15 = 13{\cdot}2$ (inches) for the rectangular beam and the tee beam of Class I and $ac = 0{\cdot}8857 \times 15 = 13{\cdot}285$ for the tee beam of Class II.

In practical design it is not always possible to adopt beam sections giving the most economical results, and comparisons on a basis such as that in the table can only be made with accuracy in cases where calculations are founded upon a uniform set of conditions for all the beams considered.

Nevertheless, investigation of the kind is desirable, not only as a matter of interest but also as an aid to engineers in the design of beams of the most favourable type possible under disadvantageous conditions, due to structural or architectural requirements.

NUMERICAL EXAMPLES
OF
ABBREVIATED STANDARD FORMULÆ
For Resistance Moments in
RECTANGULAR AND TEE BEAMS

(*See Folding Plate facing page* 166)

DATA.—b = 10″ for Rectangular Beams
b = 40″ for Tee Beams
d = 15″ for all beams
n = 5·4″ for all beams
$n_{,}$ = 0·36 for all beams
c = 600 lb./in.^{2}
t = 16000 lb./in.^{2}
m = 15

BEAMS WITH SINGLE REINFORCEMENT

Rectangular Beam.

By (S19). $r = 0·00675$; $a_{,} = (1 - ·12) = 0·88$; and $cm = (\frac{1}{2}c) = 300$ lb./in^2.

By (S1a),
$Rt = [r{\cdot}a_{,}{\cdot}t]b{\cdot}d^2 = [·00675 \times ·88 \times 16000]b{\cdot}d^2 = 95b{\cdot}d^2$

By (S2a),
$Rc = [n_{,}{\cdot}a_{,}{\cdot}cm]b{\cdot}d^2 = [·36 \times ·88 \times 300]b{\cdot}d^2 = 95b{\cdot}d^2$
$\therefore R = 95 \times 10 \times 15^2 = 213{,}750$ inch-pounds.

Tee Beam (Class I) $\theta = n$.

As calculated by (S1a) and (S2a), $Rt = Rc = 95b{\cdot}d^2$.
$\therefore R = 95 \times 40 \times 15^2 = 855{,}000$ inch-pounds.

Tee Beam (Class II) $\theta < n$.

Let $\theta = 4·5''$ and $\theta_{,} = 4·5/15 = 0·3$. Then

By (S21), $r = 0·00656$; $a_{,} = (1 - ·1143) = 0·8857$, and $cm = (1 - \theta_{,}/2n_{,})c = 350$ lb./in^2.

By (S3a)
$Rt = [r{\cdot}a_{,}{\cdot}t]b{\cdot}d^2 = [·00656 \times ·8857 \times 16000]b{\cdot}d^2 = 93b{\cdot}d^2$

By (S4a),
$Rc = [\theta_{,}{\cdot}a_{,}{\cdot}cm]b{\cdot}d^2 = [·3 \times ·8857 \times 350]b{\cdot}d^2 = 93b{\cdot}d^2$
$\therefore R = 93 \times 40 \times 15^2 = 837{,}000$ inch-pounds.

BEAMS WITH DOUBLE REINFORCEMENT

Rectangular Beam.

Let $rc = 0{\cdot}00675$ and $i_1 = 0{\cdot}1$. Then, by ($S22$), $r = 0{\cdot}00949$; by ($S25$) and ($S26$), $cs = 6500$ lb./in^2; $a_1 = (1 - {\cdot}12) = 0{\cdot}88$; $ac_1 = (1 - {\cdot}1) = 0{\cdot}9$; $cm = (\frac{1}{2}c) = 300$ lb./in^2.

By ($S5a$),

$$Rt = [r{\cdot}a_1{\cdot}t + rc(ac_1 - a_1)cs]b{\cdot}d^2 = [{\cdot}00949 \times {\cdot}88 \times 16000 + {\cdot}00675 \times {\cdot}02 \times 6500]b{\cdot}d^2 = 134{\cdot}5b{\cdot}d^2$$

By ($S6a$),

$$Rt = [r{\cdot}ac_1{\cdot}t - n_1(ac_1 - a_1)cm]b{\cdot}d^2 = [{\cdot}00949 \times {\cdot}9 \times 16000 - {\cdot}36 \times {\cdot}02 \times 300]b{\cdot}d^2 = 134{\cdot}5b{\cdot}d^2$$

By ($S7a$),

$$Rc = [n_1{\cdot}a_1{\cdot}cm + rc{\cdot}ac_1{\cdot}cs]b{\cdot}d^2 = [{\cdot}36 \times {\cdot}88 \times 300 + {\cdot}00675 \times {\cdot}9 \times 6500]b{\cdot}d^2 = 134{\cdot}5b{\cdot}d^2$$

$\therefore R = 134{\cdot}5 \times 10 \times 15^2 = 302{,}625$ inch-pounds.

Rectangular Beam, $ac = a$.

Let $rc = 0{\cdot}00675$ and $i_1 = \frac{1}{3}n_1$. Then, by ($S23$), $r = 0{\cdot}00928$; by ($S25$) and ($S26$), $cs = 6000$ lb./in^2; $a_1 = ac_1 = (1 - {\cdot}12) = 0{\cdot}88$; and $cm = 300$ lb./in.

By ($S8a$),

$$Rt = [r{\cdot}a_1{\cdot}t]b{\cdot}d^2 = [{\cdot}00928 \times {\cdot}88 \times 16000]b{\cdot}d^2 = 130{\cdot}5b{\cdot}d^2$$

By ($S9a$),

$$Rc = [(n_1{\cdot}cm + rc{\cdot}cs)a_1]b{\cdot}d^2 = [({\cdot}36 \times 300 + {\cdot}00675 \times 6000) \times {\cdot}88]b{\cdot}d^2 = 130{\cdot}5b{\cdot}d^2$$

$\therefore R = 130{\cdot}5 \times 10 \times 15^2 = 293{,}625$ inch-pounds.

Tee Beams (Class I). $\theta = n$.

As calculated by ($S5a$), ($S6a$), and ($S7a$), $Rt = Rc = 134{\cdot}5b{\cdot}d^2$.

$\therefore R = 134{\cdot}5 \times 40 \times 15^2 = 1{,}210{,}500$ inch-pounds.

Tee Beam (Class I), $\theta = n$; $ac = a$.

As calculated by ($S8a$) and ($S9a$), $Rt = Rc = 130{\cdot}5\, b{\cdot}d^2$.

$\therefore R = 130{\cdot}5 \times 40 \times 15^2 = 1{,}174{,}500$ in.-pounds.

Tee Beam (Class I), $\theta < n$.

Taking $\theta_1 = 0{\cdot}3$, $i_1 = 0{\cdot}1$, and $rc = 0{\cdot}00675$, as before, we have, by ($S28$), $r = 0{\cdot}0093$ and by ($S30$) and ($S31$), $cs = 6500$ lb./in.^{2}; $a_1 = (1 - {\cdot}1143) = 0{\cdot}8857$; $ac_1 = (1 - {\cdot}1) = {\cdot}9$; and $cm = (1 - \theta_1/2n_1)c = 350$ lb./in.^{2}.

By ($S10a$),

$$Rt = [r \cdot a_1 \cdot t + rc(ac_1 - a_1)cs]b \cdot d^2 = [\cdot 0093 \times \cdot 8857 \times 16000 + \cdot 00675 \times \cdot 0143 \times 6500]b \cdot d^2 = 132 \cdot 5 b \cdot d^2$$

By ($S11a$),

$$Rt = [r \cdot ac_1 \cdot t - \theta_1(ac_1 - a_1)cm]b \cdot d^2 = [\cdot 0093 \times \cdot 9 \times 16000 - \cdot 3 \times \cdot 0143 \times 350]b \cdot d^2 = 132 \cdot 5 b \cdot d^2$$

By ($S12a$),

$$Rc = [\theta_1 \cdot a_1 \cdot cm + rc \cdot ac_1 \cdot cs]b \cdot d^2 = [\cdot 3 \times \cdot 8857 \times 350 + \cdot 00675 \times \cdot 9 \times 6500]b \cdot d^2 = 132 \cdot 5 b \cdot d^2$$

$\therefore R = 132 \cdot 5 \times 40 \times 15^2 = 1{,}192{,}500$ inch-pounds.

Tee Beam (Class II), $\theta = n$; $ac = a$.

Taking $\theta_1 = 0 \cdot 3$, $i_1 = \left(\dfrac{3n_1 \cdot \theta_1 - 2\theta_1^2}{6n_1 - 3\theta_1}\right)$, and $rc = 0 \cdot 00675$, we have, by ($S28$), $r = 0 \cdot 00915$: and, by ($S30$) and ($S31$), $cs = 6140$ lb./in.². Then, with $ac_1 = a_1 = 0 \cdot 8857$, and $cm = 350$ lb./in.², as before –

By ($S13a$),

$$Rt = [r \cdot a_1 \cdot t]b \cdot d^2 = [\cdot 00915 \times \cdot 8857 \times 16000]b \cdot d^2 = 129 \cdot 5 b \cdot d^2$$

By ($S14a$),

$$Rc = [(\theta_1 \cdot cm + rc \cdot cs)a_1]b \cdot d^2 = [(\cdot 3 \times 350 + \cdot 00675 \times 6140) \cdot 8857]b \cdot d^2 = 129 \cdot 5 b \cdot d^2$$

$\therefore R = 129 \cdot 5 \times 40 \times 15^2 = 1{,}165{,}500$ inch-pounds.

SUMMARY OF BEAM FORMULÆ

SIMPLE WORKING FORMULÆ

FOR

BEAMS WITH FIXED ENDS AND UNIFORMLY DISTRIBUTED LOADING.

$$(R = B = Q \cdot b \cdot d^2 = W \cdot l/10.)$$

	Formula No.
$W = \dfrac{10Q \cdot b \cdot d^2}{l}$	(61)
$l = \dfrac{10Q \cdot b \cdot d^2}{W}$	(62)
$b = \dfrac{W \cdot l}{10Q \cdot d^2}$	(63)
$d = \sqrt{\left(\dfrac{W \cdot l}{10Q \cdot b}\right)}$	(64)

Rectangular Beams, $b = \frac{2}{3}d$, *or* $d = 1\frac{1}{2}b$.

$d = \sqrt[3]{\left(\dfrac{1 \cdot 5W \cdot l}{10Q}\right)}$	(64*a*)

Tee Beams, $b = \frac{2}{3}d$, *or* $d = 1\frac{1}{2}b$.

$d = \sqrt[3]{\left(\dfrac{1 \cdot 5W \cdot l}{10Q \cdot b/br}\right)}$	(64*b*)

These equations can be readily modified to suit cases where $B = W \cdot l/2$, $B = W \cdot l/4$, $B = W \cdot l/8$, and so on.

NUMERICAL EXAMPLES

SIMPLE WORKING FORMULÆ

FOR

BEAMS WITH FIXED ENDS AND UNIFORMLY DISTRIBUTED LOADING.

$(R = B = W \cdot l/10.)$

DATA.—$b = 10$ in. for Rectangular Beam, $b = 40$ in. for Tee Beam, $d = 15$ in., $l = 100$ in., $Q = 95$, as calculated on page 143.

	Formula No.
Ex. (1) *Rectangular Beam.*	
$W = 10 \times 95 \times 10 \times 15^2/100 = 21{,}375$ (lb.) .	(61)
Ex. (1*a*).—*Tee Beam.*	
$W = 10 \times 95 \times 40 \times 15^2/100 = 85{,}500$ (lb.) .	(61)
Ex. (2).—*Rectangular Beam.*	
$l = 10 \times 95 \times 10 \times 15^2/21{,}375 = 100$ (in.) .	(62)
Ex. (2*a*).—*Tee Beams.*	
$l = 10 \times 95 \times 40 \times 15^2/85{,}500 = 100$ (in.) .	(62)
Ex. (3).—*Rectangular Beam,* $b = \frac{2}{3}d$.	
$d = \sqrt[3]{[1{\cdot}5 \times 21{,}375 \times 100/(10 \times 95)]}$	
$= \sqrt[3]{3375} = 15$ (in.)	(64*a*)
$\therefore b = \frac{2}{3}d = 10$ (in.)	
Ex. (4).—*Tee Beam,* $b = \frac{2}{3}d$, $b/br = 4$.	
$d = \sqrt[3]{[1{\cdot}5 \times 85{,}500 \times 100/(10 \times 95)4]}$	
$= \sqrt[3]{3375} = 15$ (in.)	(64*b*)
$\therefore b = \frac{2}{3}d = 10$ (in.)	

MEMORANDA

CHAPTER VIII

FORMULÆ FOR BEAMS:

WEB STRESSES AND WEB REINFORCEMENT

IN addition to the principal stresses in a reinforced concrete beam, for which formulæ have been given, there exist secondary or web stresses, including horizontal and vertical shearing stresses, and tensile and compressive stresses in every diagonal direction.

As the subject of web stresses is discussed in Chapter VI, only such explanatory notes are here given as may be necessary to make clear the origin and purport of the formulæ which follow.

Basis of Formulæ. All the formulæ for web stresses and web reinforcement are based upon the assumptions, except where otherwise stated, that the main reinforcing bars are straight throughout their length, and that longitudinal or horizontal tension is resisted entirely by the main reinforcing bars.

A general formula for the calculation of shearing stress, either horizontal or vertical, as these are always of equal value, may be derived from the familiar equations for homogeneous beams of rectangular cross section—

$$S = s \cdot b \cdot D$$

and $$s = \frac{S}{b \cdot D}$$

where S = total shear at a given vertical section, s = shearing stress per unit of the area $b \cdot D$, and D = total depth of the beam.

To comply with the conditions obtaining in a reinforced concrete beam, the equation $s = \dfrac{S}{b \cdot D}$ must be altered by substituting a = arm of the resisting moment for D = total depth of the beam.

Briefly stated, the reason for this change is that as longitudinal tension is assumed to be concentrated in the main reinforcement, and transmitted thence to the centre of compression in the concrete, the horizontal and vertical shearing stresses are considered to prevail only between the centres of tension and compression, the intensity of the shearing stresses being of uniform and maximum value below the neutral axis, and of progressively decreasing value above that level.

Therefore, for the intensity of horizontal or vertical shearing stress in a reinforced concrete beam, we have the general formula

$$S = s \cdot b \cdot a \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

and

$$s = \frac{S}{b \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Suitable modifications of (65) can be made so as to provide formulæ for the calculation of web stresses and different types of web reinforcement.

BEAMS WITHOUT WEB REINFORCEMENT

Horizontal Shearing Stress Around the Longitudinal Bars.— In a beam without web reinforcement, the transference of tension from the longitudinal bars to the surrounding concrete is accompanied by the development of horizontal shearing stress, tending to destroy the grip or bond between the concrete and the bars.

In order that the bond may be maintained, g, the grip per unit area of contact surface, multiplied by As, the total superficial area of the bars per unit length of beam, must be equal to $s \cdot b$, the shearing stress distributed over a horizontal section, of a unit length of beam and of breadth b, just above the plane of the bars.

Thus,

$$g \cdot As \cdot l = s \cdot b \cdot l$$

and

$$g \cdot As = s \cdot b$$

Therefore formula (65) can be altered from

$$S = s \cdot b \cdot a$$

to

$$S = g \cdot As \cdot a \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Whence

$$g = \frac{S}{As \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$As = \frac{S}{g \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Formula (68) gives the grip stress per unit of surface area when the superficial area of the bars has been settled, and (69) gives the required superficial area in cases where the value of the grip stress has been predetermined.

GRIP OR ADHESION LENGTH OF BARS

The tensile force T required to break a round steel bar embedded in concrete without destroying the grip or adhesion bond is given by the equation

$$T = t(\tfrac{1}{4} \cdot \pi \cdot d^2) \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where

T = *Tension* (total) in pounds

t = *tensile ultimate* resistance of the steel in pounds per square inch.

π = *peripheral* ratio of a circle = ratio of circumference to diameter.

d = *diameter* of the bar in inches.

The tensile force T required to destroy the grip or adhesion bond without breaking the bar is

$$T = (l \cdot \pi \cdot d)g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

where l = grip or adhesion length of the embedded bar in inches, $l \cdot \pi \cdot d$ = surface area of the bar in square inches, and g = grip or adhesion in pounds per square inch.

Equating (a) and (b) we obtain a simple formula for finding the grip length required to make the strength of the bond equal either to the ultimate tensile resistance or to the safe working stress of the embedded bar.

Thus

$$l = (\tfrac{1}{4} \cdot t \cdot d)/g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Vertical and Horizontal Shearing Stresses.—Formula (66) is suitable as it stands for calculating the vertical or horizontal shearing stresses in a reinforced concrete beam. Such calculations are rarely, if ever, necessary to determine the safety of a beam against failure by vertical shear, for the reason that the maximum shearing stress developed in tests to destruction of reinforced concrete beams is generally found to be far less than the shearing resistance of the concrete, beams so tested failing by diagonal tension or some other stress. In ordinary practice, the chief use of (66) is in the computation of vertical shearing stress, for the purpose of measuring or estimating the amount of diagonal tension developed in the web of a beam.

Diagonal Tension.—Although it is assumed in all formulæ based on the straight-line theory of flexure that longitudinal tension is resisted entirely by the principal reinforcement of a reinforced concrete beam, the assumption is not true, and cannot be true unless the resistance of the concrete has been entirely destroyed by tensile failure.

While this assumption has very little influence on the accuracy of the formulæ given for the calculation of vertical and horizontal shearing stresses, and merely leads to a small error on the side of safety in the calculation of resisting moments, the error would be on the other side if the same assumption were adopted in estimating the diagonal tensile stress in the web of a reinforced concrete beam.

The equation given in text-books on applied mechanics for the diagonal stress resulting from the combination of tensile or compressive stress with shearing stress, may be written for diagonal tensile stress

$$td = \tfrac{1}{2}tc + \sqrt{(\tfrac{1}{4}tc^2 + s^2)}$$

Here, td = diagonal tensile stress, tc = longitudinal or horizontal tensile stress in the concrete, and s = vertical shearing stress. The direction of the maximum diagonal stress makes an angle with the horizontal equal to half the angle the cotangent of which is $\frac{1}{2}tc/s$. Otherwise expressed

$$a = \tfrac{1}{2}\angle\ cot(\tfrac{1}{2}t/s)$$

A few simple calculations will show that the intensity and direction of the maximum diagonal tensile stress must be affected considerably by the amount of horizontal tensile stress in the concrete of a beam.

Thus, if

$tc = s \times 0{\cdot}0$	$td = s$	and $\alpha = 45^\circ 00'$
$tc = s \times 1$	$td = 1{\cdot}6s$	and $\alpha = 31^\circ 43'$
$tc = s \times 1{\cdot}5$	$td = 2{\cdot}0s$	and $\alpha = 26^\circ 34'$
$tc = s \times 2$	$td = 2{\cdot}4s$	and $\alpha = 22^\circ 30'$

Consequently, diagonal tensile stress may easily attain an intensity exceeding the tensile resistance of the concrete, thereby causing a diagonal tension failure, formerly described as a "shear" failure, in a beam where there is no web reinforcement.

Since the horizontal tensile stress in the concrete is practically indeterminable, the general practice is to employ the vertical shearing stress, calculated by (66), as a measure of the diagonal tensile stress, bearing in mind the fact that this is greater than the vertical shearing stress, probably varying from s up to $2s$ in a beam of average design.

BEAMS WITH WEB REINFORCEMENT

The diagonal force to be resisted by stirrups and other web members is tension, although shear is taken as its measurement, and the stress developed in the steel composing the web members is tensile and not shearing stress, as often stated, or as implied by some formulæ for the design of web reinforcement.

In the following equations, the symbol t is used to denote the actual nature of the stress in the steel, instead of adopting the somewhat confusing practice of denoting tensile stress by the symbol ss, which means shearing stress in steel.

Vertical Stirrups.—Formula (65) can be modified as shown below for the calculation of vertical stirrups or analagous web members, diagrammatically represented in Fig. 74.

Let Td be the total diagonal tension, represented by N times the total vertical shear, or $Td = N{\cdot}S$.

Then, replacing s by t = tensile stress, and $a \cdot b$ by Aw = sectional area of web reinforcement and adding the spacing or pitch ratio $\frac{a}{p}$ where a = arm of resistance moment, and

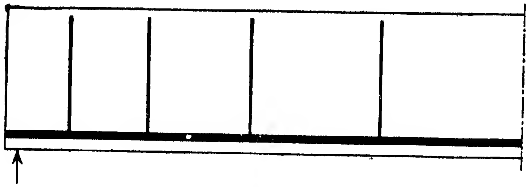


FIG. 71

p = pitch or longitudinal distance between the stirrups or groups of stirrups, we obtain

$$Td = N \cdot S = t \cdot Aw\left(\frac{a}{p}\right). \quad . \quad . \quad . \quad (71)$$

Whence

$$t = N \cdot \frac{S \cdot p}{Aw \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$Aw = N \cdot \frac{S \cdot p}{t \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$p = \frac{t \cdot Aw \cdot a}{N \cdot S} \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

For reasons already stated, the actual value to be assigned to the numerical factor N cannot be determined precisely, but the results of experiments and practical experience show that in average practice the value $N = 1{\cdot}5$ should furnish an ample assurance of safety.

If we take $Td = 1{\cdot}3\, S$, (71) gives the same results as those calculated by the equation : $S = \frac{ss \cdot Aw \cdot a}{p}$ constituting the basis of the formulæ for vertical stirrups recommended by the R.I.B.A. Joint Committee, the Inst. C.E. Committee, and others, and (71) could be expressed in identical terms

by omitting the factor N and using the symbol ss — shearing stress in steel, to denote tensile stress, the value of ss being taken at $\frac{t}{1{\cdot}3}$.

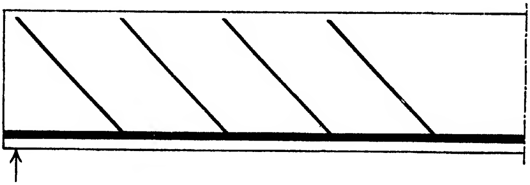


FIG. 75

Diagonal Stirrups. The uncertainties and variable conditions attaching to the design of vertical stirrups are still more pronounced in the case of diagonal stirrups and other web members.

The efficiency of diagonal web stirrups depends very largely upon the manner in which they are attached to the main longitudinal bars, upon the adequacy of the bond or grip between the concrete and the stirrups, and upon the angular direction of the latter in relation to that of the diagonal tensile stress.

Let us assume that practical conditions have been complied with, that the diagonal tensile stress acts at an angle of 45°, and that the stirrups are placed at the same angle, as in Fig. 75. Then, as the stirrups act at the same inclination as the stress to be resisted, they are theoretically more efficient than vertical stirrups acting obliquely to the direction of the diagonal stress.

By the principles of mechanics, the theoretical efficiency of diagonal stirrups at an angle of 45° is greater than that of vertical stirrups in the ratio $\sqrt{2} : 1$, or the equivalent ratio $1 : \sin 45°$.

Consequently, the sectional area of steel in the stirrups can be reduced proportionately, and modified on this basis, (71) becomes

$$Td = \frac{N{\cdot}S}{\sqrt{2}} = \frac{t{\cdot}Aw{\cdot}a}{p} = N{\cdot}S{\cdot}\sin 45° = \frac{t{\cdot}Aw{\cdot}a}{p} \quad . \qquad (75)$$

Having assumptions of somewhat problematical nature for its basis, this equation need not be complicated by the presence of inconvenient decimal fractions. Therefore, instead of retaining either $\frac{1}{\sqrt{2}} = 1·4142$ or $\sin 45° = 0·7071$, and $N =$ either 1·5 or $1·\dot{3}$, giving $\sin 45° \times N = 1·0606$ in one case, and 0·9428 in the other, we may very well accept the compromise: $\sin 45° \times N = 1$, and write (75) for $a = 45°$ in the simple form

$$Td = S = \frac{t \cdot Aw \cdot a}{p} \quad . \quad . \quad . \quad . \quad . \quad (76)$$

$$t = \frac{S \cdot p}{Aw \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

$$Aw = \frac{S \cdot p}{t \cdot a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

$$p = \frac{t \cdot Aw \cdot a}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

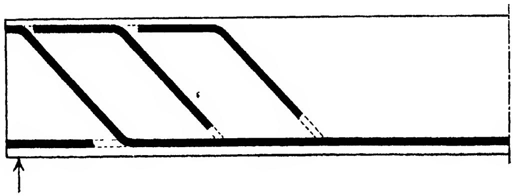


FIG. 76

As in the case of (71), the first form of (75) can be expressed in the same way as the basic equation

$$\frac{S}{\sqrt{2}} = \frac{ss \cdot Aw \cdot a}{p}$$

recommended by the R.I.B.A. Joint Committee and the Inst. C.E. Committee, by the omission of N, and the substitution of ss for t. Formula (75) gives the same results

as the equation mentioned, and its simplified form (76) requires a slightly increased area of steel.

Bent-up Longitudinal Bars.—Web reinforcement formed by bending up some or all of the main longitudinal bars of a beam at an angle of about 45° may be calculated by (77) to (79) where the bars are bent up at different points, so as to constitute a series of successive diagonal members as in Fig. 76; or by (77) and (78) where the bars are bent up at one point only, as in Fig. 77.

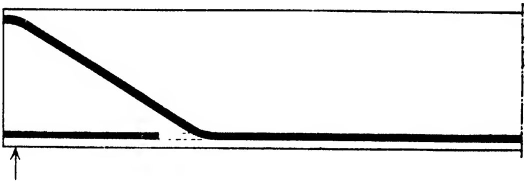


FIG. 77

Owing to the very indeterminate character of the stress distribution in beams with bars bent up to form angles of materially less than 45°, no precise methods of calculation are available. It is evident, however, that the area of steel, as calculated by (78), should be increased as the angle made by the bent up bars is decreased, a simple rule for general guidance being—

$$Aw = (S \cdot p/l \cdot a) \cdot \frac{\sin 45^\circ}{\sin a} \qquad (80)$$

where $S \cdot p/l \cdot a$ = sectional area of steel as given by (78), Aw = required sectional area of steel as given by (80), and a = angle at which the bars are bent up from the horizontal.

In beams where some of the longitudinal bars are bent up at one point only, the value of the bent up portions as web reinforcement should be calculated, and any further web reinforcement required should be added in the form of vertical stirrups.

Bent-up Bars with Vertical Stirrups.—In beams where these two classes of web reinforcement are employed in

combination, the relative efficiency of the bent-up bars and the vertical stirrups cannot be determined with accuracy, but there are reasons for the view that the full tensile resistance of the steel cannot be developed simultaneously in the two forms of web reinforcement.

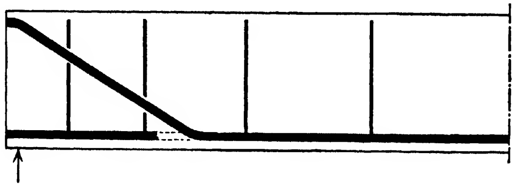


FIG. 78

Consequently, it would not be good practice to calculate the amount of diagonal tension taken by bent-up bars, and to proceed on the assumption that the whole of the remainder would be taken by vertical stirrups (*see* Fig. 78) calculated by (72) to (74).

One method affording security against inadequate resistance is to calculate the effect of bent-up bars by (76) to (80), and then to provide, for resistance to the remainder of the stress, from $1\frac{1}{2}$ times to twice the sectional area of steel given by the use of (73).

Another method with the same object in view is one which has been employed for many years with satisfactory results. In this, if the bent-up portions of the bars are found by calculation to be capable of resisting half, or more than half, the calculated shearing force, vertical stirrups are provided to resist the remainder, taken at not less than half the original amount. In case the bent-up bars are found to be insufficient for resistance to half the shearing force, the sectional area of steel in the stirrups is increased accordingly.

PROPORTIONING AND SPACING WEB REINFORCEMENT

Beams Under Concentrated Centre Load.—In the case of a beam freely supported at the ends with a concentrated load at the centre of the span, the shearing force diagram

is of rectangular form, as in the lower part of Fig. 79. Consequently the total amount of steel required as web reinforcement in the form of vertical or diagonal stirrups should be equally distributed in an appropriate number of stirrups, or sets of stirrups, spaced at equal intervals apart in the length of the beam.

In accordance with the principles of mechanics, the bending moment at any section of a beam is equal to the area of the shearing force diagram up to that point measured on the length-load scale adopted.

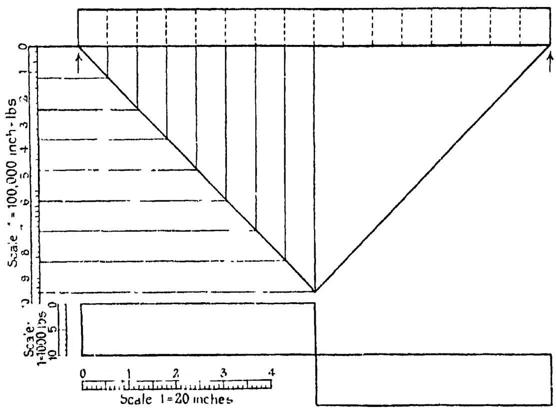


FIG. 79

This interesting connection between bending moment and shearing force diagrams is illustrated in Fig. 79.

In Fig. 79, the scale of bending moments is divided into eight equal parts and the horizontal lines from the nine corresponding points intersect the bending moment diagram at points whence vertical lines are drawn indicating the positions of the stirrups in the beam above.

Beams Under Uniformly-Distributed Load.—It is evident at first sight that with a bending moment diagram such as that in Fig. 79, the equal division of the scale of bending moments must lead to equal spacing of the stirrups.

On the other hand, with bending moment diagrams of parabolic outline, as in Figs. 80 and 81, equal division of the bending moment scale leads to unequal spacing of the

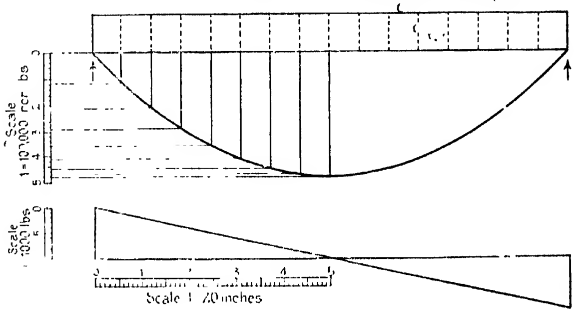


FIG. 80

stirrups; and, conversely, equal spacing of the stirrups can only be secured by unequal division of the bending moment scale.

Fig. 80 shows the connection between bending moment and

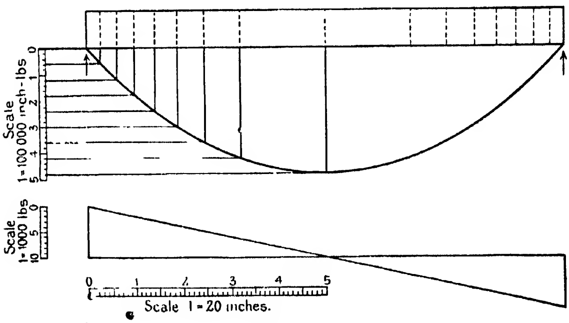


FIG. 81

shearing force diagrams for a beam under uniformly-distributed loading, and if employed as an aid to the proportioning of equally spaced stirrups, it would enable the user to ascertain quite readily the shearing forces at the various points along the beam. However, as it would be both

troublesome and costly to change the sectional area of the stirrups every few inches along a beam, this method of procedure cannot be recommended.

Fig. 81 also shows the relation between bending moments and shearing forces, and illustrates a very convenient method of determining the spacing of stirrups in accordance with variations of shearing force from point to point along a beam.

The required sectional area of steel for the first set of stirrups can be calculated by the formulæ already given, and the sectional area so determined will be employed for the remaining sets between each support and the centre of the span.

The first set of stirrups should be at a distance from either support not more than ($\frac{1}{2}a$), or one-half the length of the arm of leverage of the internal forces, and the remaining sets can be spaced at proper intervals apart by the aid of a diagram such as Fig. 81.

The required number of sets of stirrups is governed by the ratio between the length and depth of the beam, for in beams of relatively short span under heavy loads the shearing force is a very important factor, and unless the web reinforcement is closely spaced and of adequate sectional area there will be serious risk of failure by diagonal tension, for which vertical shear is generally employed as a convenient measure.

SUMMARY OF BEAM FORMULÆ

(WEB STRESSES AND REINFORCEMENT)

FORMULÆ FOR BEAMS OF ALL CLASSES

When used for Tee beams, equations (65) and (66) must be modified by the substitution of *br* for *b*.

	Formula No.
Vertical or Horizontal Shear.	
$S = s \cdot b \cdot a$	(65)
$s = S/b \cdot a$	(66)
Horizontal Shear (Grip Stress).	
$S = g \cdot As \cdot a$	(67)
$g = S/As \cdot a$	(68)
$As = S/g \cdot a$	(69)
$l = (\frac{1}{4}t \cdot d)/g$	(70)
Diagonal Tension.	
Vertical Stirrups.	
$Td = N \cdot S = t \cdot Aw \cdot a/p$	(71)
$t = N \cdot (S \cdot p/Aw \cdot a)$	(72)
$Aw = N \cdot (S \cdot p/t \cdot a)$	(73)
$p = t \cdot Aw \cdot a/N \cdot S$	(74)
Diagonal Stirrups.	
$Td = N \cdot S/\sqrt{2} = t \cdot Aw \cdot a/p$	(75)
$Td = S = t \cdot Aw \cdot a/p$	(76)
$t = S \cdot p/Aw \cdot a$	(77)
$Aw = S \cdot p/t \cdot a$	(78)
$p = t \cdot Aw \cdot a/S$	(79)
Bent-up Bars.	
$Aw = (S \cdot p/t \cdot a) \cdot (\sin 45^\circ/\sin \alpha)$	(80)

NUMERICAL EXAMPLES

DATA.—$As = 16$ sq. in., $a = 13{\cdot}2$ in., $b = br = 10$ in., $g = 100$ lb./in.², $N = 1{\cdot}5$, $p = 10$ in., $s = 60$ lb./in.², $t = 16000$ lb./in.².

	Formula No.
Vertical or Horizontal Shear.	
$S = 60 \times 10 \times 13{\cdot}2 = 7{,}920$ (lb.)	(65)
$s = 7920/(10 \times 13{\cdot}2) = 60$ (lb./in.²)	(66)
Horizontal Shear (Grip Stress).	
$S = 100 \times 16 \times 13{\cdot}2 = 21{,}120$ (lb.)	(67)
$g = 21120/(16 \times 13{\cdot}2) = 100$ (lb./in.²)	(68)
$As = 21120/(100 \times 13{\cdot}2) = 16$ (sq. in.)	(69)
Grip Length for bar, $d = 1$ in.	
$l = (\frac{1}{4} \times 16000 \times 1)/100 = 40$ (in.)	(70)
Diagonal Tension.	
Vertical Stirrups.	
$Td = 1{\cdot}5 \times 21120 = 31{,}680$ (lb.)	
$= (16000 \times Aw \times 13{\cdot}2)/10 = 21{,}120Aw$	(71)
$\therefore Aw = 1{\cdot}5$ sq. in.	
$t = 1{\cdot}5[21120 \times 10/(1{\cdot}5 \times 13{\cdot}2)] = 16000$ (lb./in.²)	(72)
$Aw = 1{\cdot}5 \times [21120 \times 10/(16000 \times 13{\cdot}2)] = 1{\cdot}5$ (sq. in.)	(73)
$p = (16000 \times 1{\cdot}5 \times 13{\cdot}2)/(1{\cdot}5 \times 21{,}120) = 10$ (in.)	(74)
Diagonal Stirrups (45°).	
$Td = 1{\cdot}5 \times 21120/1{\cdot}414 = 22{,}404$ (lb.)	
$= (16000 \times Aw \times 13{\cdot}2)/10 = 21{,}120Aw$	(75)
$\therefore Aw = 1{\cdot}06$ sq. in.)	
$Td = 21120$ lb. $= 21120Aw$ ($\therefore Aw = 1$ sq. in.)	(76)
$t = 21120 \times 10/(1 \times 13{\cdot}2) = 16000$ (lb./in.²)	(77)
$Aw = 21120 \times 10/(16000 \times 13{\cdot}2) = 1$ (sq. in.)	(78)
$p = (16000 \times 1 \times 13{\cdot}2)/21120 = 10$ (in.)	(79)
Bent-up Bars (30°).	
$Aw = 21120 \times 10/(16000 \times 13{\cdot}2)\ ({\cdot}7071/{\cdot}5000)$	
$= 1{\cdot}414$ (sq. in.)	(80)

MEMORANDA

CHAPTER IX

FORMULÆ FOR COMPRESSION MEMBERS

THE formulæ given in this chapter are applicable to columns, pillars, struts, and other compression members, whether placed vertically or at any inclination to the horizontal.

With the object of avoiding unnecessary repetitions, the familiar term "column" is employed to denote a typical form and not all forms of compression members, and as a matter of convenience, the chapter is divided into two parts relating to (I) Concentric Loading, and (II) Eccentric Loading.

At the end of this chapter, summaries of formulæ are given for both conditions of loading, together with numerical examples. A selection of Standard Formulæ will be found in the folding plate facing p. 202.

(I) COLUMNS UNDER CONCENTRIC LOADING

Basis of Formulæ.– Considering first the case of a plain concrete column of uniform cross section, let P = total *pressure* applied, A = *area* of the column at any horizontal section, and c = *compressive* stress per unit area in the *concrete*.

Then

$$P = c \cdot A \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Let us now assume that vertical steel bars have been embedded in the column, displacing an amount of concrete proportionate to Av = sectional *area* of the *vertical* steel.

It is evident that the application of pressure over the whole sectional area of the reinforced column must produce equal diminution of length in the two materials. Consequently, as, by Hooke's law, the stress developed in the steel must be proportional to the actual amount of strain produced, the strength contributed by the reinforcing bars, when working in combination with the surrounding concrete,

cannot be equal to that which might be developed if the steel were employed independently.

NOTATION

(Other symbols will be found in succeeding paragraphs.)

A = *Area* (total) of a column, or other compression member.
An = *Area* (*net*) of the concrete = $A - Av$.
Av = *Area* of the *vertical* steel bars.
c = *compressive* stress in the *concrete*.
Ec = *Elastic* modulus of *concrete*.
Es = *Elastic* modulus of *steel*.
l = *length* (original).
λ = *length* increment or decrement due to stress.
m = *modular* ratio = Es/Ec.
P = *Pressure* (total).
Pc = *Pressure* (total) on the *concrete*.
Ps = *Pressure* (total) on the *steel*.

The intensity of the compressive stress on the concrete is

$$c = \frac{Pc}{An}$$

$$= \frac{Pc}{(A - Av)}$$

The compressive strain on the concrete = λ/l.

The elastic modulus for concrete in compression is

$$Ec = \frac{stress\ intensity}{strain\ intensity}$$

or

$$Ec = \frac{Pc/(A - Av)}{\lambda/l}$$

Whence we may derive the equation

$$Pc = Ec(\lambda/l)(A - Av)$$

Thus, in accordance with Hooke's law, the proportion of the total pressure taken by the concrete is

$$Pc = Ec \cdot \frac{\lambda}{l}(A - Av)$$

and the proportion taken by the steel is

$$Ps = \left(Es \cdot \frac{\lambda}{l}\right) Av$$

$$= Es \cdot \frac{\lambda}{l} \cdot Av \quad . \quad . \quad . \quad . \quad (c)$$

Therefore as $P = Pc + Ps$ we have

$$P = Ec \cdot \frac{\lambda}{l}(A - Av) + Es \cdot \frac{\lambda}{l} \cdot Av \quad . \quad . \quad . \quad (d)$$

Since $\dfrac{stress}{strain} = E$, we can write

$$stress = E \times strain.$$

In our particular case we have $c = Ec \cdot \dfrac{\lambda}{l}$

and as $Es/Ec = m$, the modular ratio, we can abbreviate equation (d) as follows.

Thus, instead of $Ec \cdot \dfrac{\lambda}{l}$ we may write c, instead of Es we can use $m \cdot Ec$, and instead of $m \cdot Ec \cdot \dfrac{\lambda}{l}$ we may use $m \cdot c$

Therefore, commencing with

$$P = Ec \cdot \frac{\lambda}{l}\left(A - Av\right) + Es \cdot \frac{\lambda}{l} \cdot Av$$

we obtain the following successive forms of the equation—

$$P = c(A - Av) + m \cdot Ec \cdot \frac{\lambda}{l} \cdot Av$$

$$= c(A - Av) + m \cdot c \cdot Av$$

Expanding the terms within the brackets we obtain

$$P = c \cdot A - c \cdot Av + m \cdot c \cdot Av$$

and by re-arrangement of the terms we get

$$P = c(A - Av + m \cdot Av)$$

$$= c(A + m \cdot Av - Av)$$

$$= c[A + (m - 1)Av]. \quad . \quad . \quad . \quad . \quad (e)$$

This equation is the basis of the column formulæ in general use, the terms in which it is expressed varying slightly, and the notation employed varying considerably.

Equivalent Area Defined.—In a column section of area A, the area Av of the vertical reinforcing bars is equivalent in resistance to an area $m \cdot An$ of concrete, and the column section is equivalent to an imaginary section of the area $[A + (m-1)Av]$, termed the equivalent area Ae.

Short Columns.—As ordinarily applied, the term *short column* means a column where the unsupported length is not greater than from 15 to 25 times the diameter or the least transverse dimension, the ratio $l/d = 18$ being a safe value for general use.

Equation (e) can be modified as shown below to provide formulæ for the design of columns reinforced (a) with vertical bars only, (b) with vertical bars and transverse ties or links, and (c) with spiral binding or hooping.

It should be borne in mind, however, that vertical bars placed near the surface of the concrete and not tied laterally would tend to buckle and to burst or spall the concrete.

(a) COLUMNS WITH VERTICAL BARS ONLY.—In the first place, equation (e) may be written

Total Pressure

$$P = c[A + (m-1)Av] \quad . \quad . \quad . \quad . \quad (81)$$

If we use r to represent the ratio Av/A, we have $Av = r \cdot A$, and obtain the equation—

$$P = c[A + (m-1)r \cdot A]$$
$$= [1 + (m-1)r]c \cdot A \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Unital Pressure

Denoting *unital pressure* by $p = P/A$, we get

$$p = c[A + (m-1)r \cdot A]/A$$
$$= [1 + (m-1)r]c \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Ratio of Vertical Reinforcement

$$r = \frac{p - c}{(m-1)c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Compressive Stress in Concrete

$$c = \frac{p}{1 + (m - 1)r} \qquad (85)$$

(*b*) COLUMNS WITH VERTICAL BARS AND TRANSVERSE TIES.—For computing the resistance added by transverse ties, equation (81), requires a supplemental expression for the effect of the pitch and amount of steel in the ties.

Let pb = *pitch*, or vertical spacing, of the *binding* ties and d = *diameter* of the column.

Then for the spacing factor we have $s = \dfrac{pb}{d}$

Further, if Vb = *volume* of steel in the *binding* ties per unit of length of column, and V = *volume* of the column per unit length, we have for the *volumetric* ratio of transverse reinforcement, $V_r = Vb/V$.

Therefore, the increased resistance due to the ties is equal to $c[1 + s{\cdot}V_r]$. Incorporating this expression in (82) and (83), we obtain

$$P = (1 + s{\cdot}V_r)\,[1 + (m-1)r]c{\cdot}A \qquad (86)$$

and

$$p = (1 + s{\cdot}V_r)\;[1 + (m-1)r]c \qquad (87)$$

The values of s in Table V are based upon the recommendations in the Report of the French Commission du Ciment Armé, with suitable modifications, the maximum value for $pb = 0{\cdot}33d$ having been slightly increased, and the minimum value for $pb = d$ reduced to zero, to provide for the effect of the differences between $(m-1)$, here taken at 14, and the varying values of the factor m' in the French equation, which is given as formula (88).

It should be noted that the French Commission define d as the outside diameter of the column, and V as the volume of the entire column per unit length, and do not restrict measurements to the concrete enclosed by the reinforcement as proposed in some interpretations.

TABLE V

VALUES OF SPACING FACTOR FOR RECTILINEAR TIES

Pitch of Binding (pb) in terms of Column Diameter (d)	Spacing Factor (s)
0·33 d	16·6 (*maximum*)
0·4 d	15
0·5 d	12·5
0·6 d	10
0·7 d	7·5
0·8 d	5
0·9 d	2·5
1·0 d	0 (*minimum*)

The formula of the Commission du Ciment Armé, and the rules for its employment, are as follows–

$$P = (1 + s \cdot V_t)c(A + m' \cdot Av) \quad . \quad . \quad . \quad (88)$$

As stated in the Report, m' is not the same as m, the ordinary modular ratio, but is a modified modular ratio or factor derived from the results of column tests. For rectilinear ties, it has a maximum value of 15 when the diameter of the vertical bars does not exceed $\frac{1}{20}$th of the column diameter d, and the pitch of the binding is less than $pb = 0·33d$; and has a minimum value of 8 when the diameter of the vertical bars exceeds $\frac{1}{10}$th of the column diameter d, and the pitch of the binding is $pb = d$.

The spacing factor s for transverse ties has a maximum value of 15 for $pb = 0·23d$ and a minimum value of 8 for $pb = d$.

(*c*) COLUMNS WITH VERTICAL BARS AND SPIRAL BINDING.—By the adoption of suitable values for the spacing factor, formulæ (86), (87), and (88) are applicable to the design of columns with helical binding.

The values of s in Table VI are practically identical with those given in the Report of the French Commission, the factor m' having a constant value so near to that of $(m - 1)$, taken at 14, as to render any appreciable modification unnecessary. The only alteration made is the substitution of 16 for 15, as the minimum value, in order to establish a uniform relationship in the figures.

TABLE VI

VALUES OF SPACING FACTOR FOR SPIRAL BINDING

Pitch of helical binding (pb) in terms of Column Diameter (d)	Spacing Factor (s)
0·2 d	32 (*maximum*)
0·3 d	24
0·4 d	16 (*minimum*)

For helical binding, the value of m' may be taken at 15 in formula (88), the values of s applying so long as the compressive stress in the concrete does not exceed 711 lb. per square inch, and providing that not fewer than six vertical bars are employed, having a sectional area equal to not less than 0·5 per cent. of the area of the concrete enclosed by the helical coils, and a volume of not less than one-third that of the helical binding.

Another condition is that the working stress in the concrete shall not exceed 60 per cent. of the ultimate strength of plain concrete, however great the proportion of reinforcement, or the value of $(1 + s \cdot V_t)$.

Owing to the inclusion of the two variable factors s and m', the French formula and rules are somewhat complicated and do not lend themselves to the tabulation of values for the factors mentioned. Moreover, in the opinion of the author, they over-estimate the value of transverse reinforcement in the form of helical binding in practical construction. Although this particular type of reinforcement undoubtedly adds greatly to the ultimate strength of a reinforced concrete compression member, it has comparatively little influence when ordinary working stresses are in question, and consequently ought not to be regarded as being more efficient than other forms of transverse reinforcement in everyday practice.

The rules for the design of pillars and columns in the R.I.B.A. Report and the L.C.C. Regulations are largely copied from the French rules, but, as represented in Fig. 82, they penalize rectilinear binding to a very serious extent, for

which there is no justification in the Report of the French Commission.

As shown in the diagram the spacing factor values for helical binding are practically identical under the French, the L.C.C., and the R.I.B.A. rules, while there are wide divergencies between the spacing factor values given in the

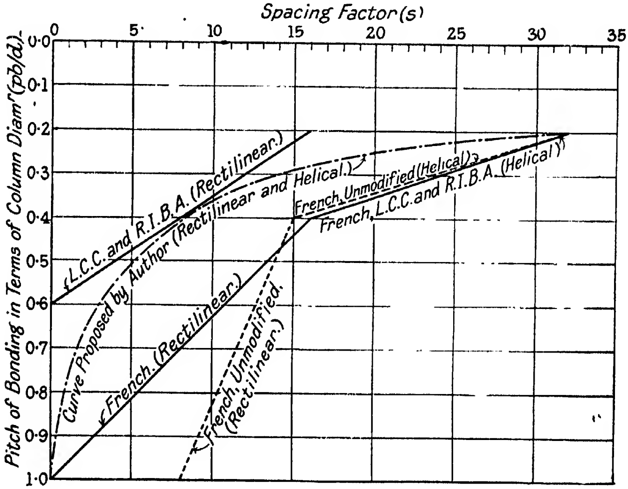


FIG 82

The French curves are based upon d overall diameter of the column and, with the exception of the two "unmodified" curves, are in accordance with Tables V and VI. The L.C.C. and R.I.B.A curves are based upon d diameter of the core of the column.

French Report for rectilinear binding and those adopted in the L.C.C. Regulations and the R.I.B.A. Report.

Moreover, although the French rules permit the whole sectional area of a column to be taken into account in the calculation of effective resistance, the L.C.C. and the R.I.B.A. rules regard only the core, or part of the section enclosed by transverse reinforcement, as being effective.

This restriction is necessary in the case of a formula for hooped columns by Considère and given in the French Report, but is unnecessary, and is not recommended in the

same Report, for use in connection with the formula and rules of the Commission du Ciment Armé.

The "dot-and-dash" curve in Fig. 82 is one proposed by the author, in place of the other curves in the same diagram, for use in the design of columns with any effective form of transverse binding. Starting at $s = 0$ with the modified French curve for rectilinear binding, it meets the L.C.C. and R.I.B.A. curve at $s = 8$, and stops at $s = 32$, where it meets the French, L.C.C., and R.I.B.A. curves for helical binding.

Long Columns.—The permissible load for a long column is determined by modifying the value of P, as calculated for a short column, either by the employment of a suitable formula, or by means of a load-reduction scale.

In either event, provision must be made for the variation of permissible load with end fixity conditions.

Where a formula is used, such provision is made by the insertion of an *end-fixity* factor (cf), and where a load-reduction scale is adopted, the influence of different end-fixity conditions is provided for by substituting an equivalent or *virtual length* (lv) in calculating the ratio l/d = *length/diameter*, or the ratio l/g = *length/gyration radius.*

Table VII gives end-fixity factors for four methods of fixing the ends of a column, as recommended by the French Commission, and the corresponding virtual length for each method.

TABLE VII

END FIXITY FACTORS AND VIRTUAL LENGTHS FOR LONG COLUMNS

Method of Fixing the Ends.			End-Fixity Factor cf	Virtual or Equivalent Length lv
No.	Ends fixed	How Fixed		
1	Both	In position and direction	$\frac{1}{4}$*	$lv = 1l$*
2	One	In position and direction		
	One	In position only	$\frac{1}{2}$†	$lv = 1{\cdot}4l$†
3	Both	In position only	1	$lv = 2l$
4	One	In position and direction, other end free	4	$lv = 4l$

* If the fixity of one end is imperfect, increase cf to $\frac{3}{8}$ and lv to $1\frac{1}{4}l$; if the fixity of both ends is imperfect, increase cf to $\frac{2}{3}$ and lv to $1\frac{5}{8}l$.

† If the fixity of the ends is imperfect, increase cf to $\frac{3}{4}$, and lv to $1\frac{3}{4}l$.

Gordon's Formula.—This familiar equation is usually written

$$P = \frac{fA}{1 + a\frac{l^2}{d^2}}$$

Employing Standard Notation the equation becomes

$$P = \frac{c \cdot A}{1 + n \cdot \frac{l^2}{d^2}}$$

where

A = Area of the column.
c = *crippling* stress or *compressive* stress intensity.
d = Least *diameter* of the column.
l = *length* of the column.
n = A *numerical* constant, or constant number.
P = Total *pressure* on the column.

For reinforced concrete $c \cdot A$ corresponds with $c \cdot Ae$ and equals $c[A + (m-1)Av]$.

Now $c \cdot A$ or $c[A + (m-1)Av]$ represents the total pressure on a *short* column, and if we use Ps to represent the total pressure on a *short* column and P to represent the total pressure on a *long* column we obtain the equation

$$P = \frac{Ps}{1 + n \cdot \frac{l^2}{d^2}}$$

The numerical constant n depends both upon the properties of the material and upon the properties of the section, and for reinforced concrete can be replaced by two factors, K for the kind of material and F for the form of cross section.

Thus modified, and with the insertion of ef, the end-fixity factor, the formula becomes—

$$P = \frac{Ps}{1 + F \cdot K \cdot ef \cdot \frac{l^2}{d^2}} \quad . \quad . \quad . \quad . \quad . \quad (89)$$

The probable value of F can be represented by the ratio $Ae{\cdot}d^2/Ie$ where Ie = inertia moment of the equivalent section.

Therefore

$$F = \frac{Ae{\cdot}d^2}{Ie} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (90)$$

Adopting Euler's theory, the value of K, for a column with both ends fixed in position only, is derived from the expression $\frac{\pi^2{\cdot}Ec}{cu}$, where cu = *ultimate compressive* strength of the concrete.

Therefore,

$$K = \frac{cu}{\pi^2{\cdot}Ec} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (91)$$

Rankine's Formula.—Introduced as an improved form of Gordon's equation, this formula is generally written

$$P = \frac{fA}{1 + a\frac{l^2}{g^2}}$$

Expressed in Standard Notation the formula is

$$P = \frac{c{\cdot}A}{1 + n{\cdot}\frac{l^2}{g^2}}$$

where n is a *numerical* constant depending upon the properties of the material alone and g = *gyration* radius $= \sqrt{(Ie/Ae)}$, depending upon the form of the section.

Modified as described in the preceding paragraph, with the insertion of K, see equation (91), and ef = *end-fixity* factor, see Table VII, the formula becomes

$$P = \frac{Ps}{1 + K{\cdot}ef{\cdot}\frac{l^2}{g^2}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (92)$$

Load-Reduction Scales.—Owing to the lack of experimental data as to the behaviour of reinforced concrete long

columns, Gordon's and Rankine's formulæ should only be used for purposes of comparison and for general guidance in the preparation of load-reduction scales.

The rules proposed by various authorities differ considerably, some denoting excessive caution, and most of them reducing the load in a straight line instead of following the curve shown by the results of the elastic theory.

Fig. 83 is a diagram containing curves which represent

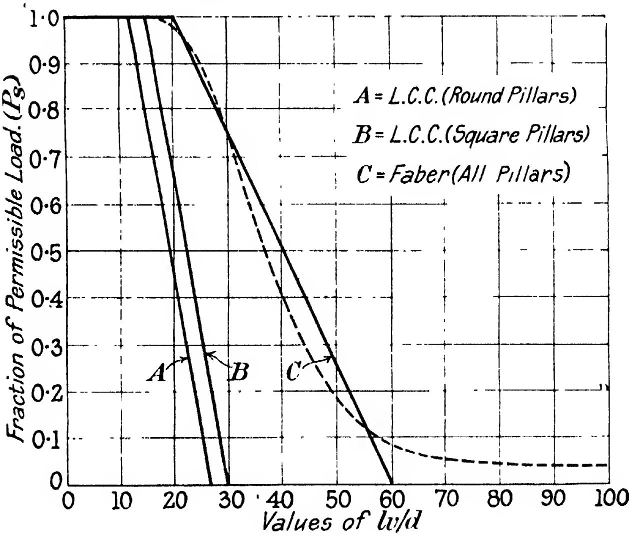


FIG. 83

different views as to the progressive reduction of the permissible short column load (Ps) with progressively increasing values of the ratio of lv/d.

Curves A and B, illustrating the London County Council rules, are of distinctly conservative nature, particularly as in this case d = diameter of the core and not the diameter of the column.

Curve C is one recommended in *Reinforced Concrete Design*, by Faber and Bowie, and the broken line curve is one typical of the curves obtainable by the use of Gordon's formula.

The author recommends the adoption of a load-reduction curve of this form for two reasons: (1) that it avoids the sudden change of direction at the top of A, B, and C, and (2) that it obviates the sudden extinction of permissible load at the bottom of the same curves.

Fig. 84 is a diagram based on the ratio lv/g, curve A representing the rules of the London County Council, and curve B showing the relative values of permissible loads for short and long columns, as proposed by Mr. Ewart S. Andrews. The latter follows approximately the typical form given by Rankine's formula, but would be improved by eliminating the sharp angle at the top, as shown by the dotted curve in the diagram.

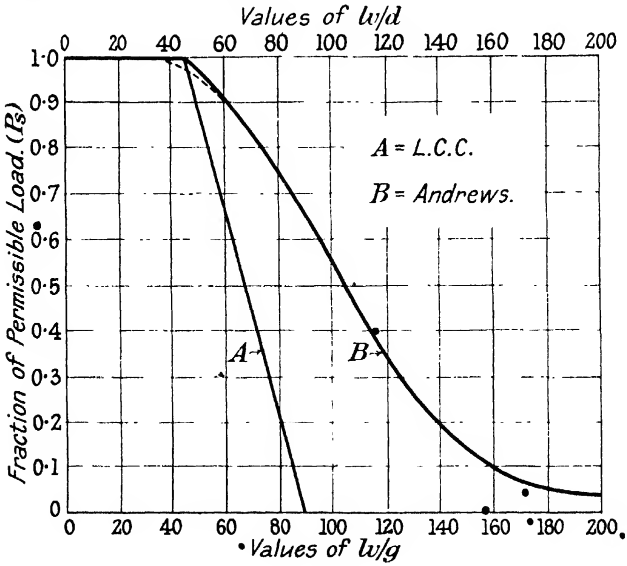


FIG. 84

A curve such as B in its amended form is preferable to a straight-line curve such as A for the reasons stated above and it may be added that the L.C.C. rules are unnecessarily conservative.

SUMMARY OF COMPRESSION MEMBER FORMULÆ

MEMBERS UNDER CONCENTRIC LOADING

SHORT COLUMNS.	Formula No.
Vertical Reinforcement only.	
$P = c[A + (m-1)Av]$	(81)
$P = [1 + (m-1)r]c \cdot A$	(82)
$p = [1 + (m-1)r]c$	(83)
$r = \dfrac{p-c}{(m-1)c}$	(84)
$c = \dfrac{p}{1+(m-1)r}$	(85)
Vertical and Transverse Reinforcement.	
$P = (1 + s \cdot V_t)\,[1 + (m-1)r]c \cdot A$	(86)
$p = (1 + s \cdot V_t)\,[1 + (m-1)r]c$	(87)
(French Commission Formula.)	
$P = (1 + s \cdot V_t)c \cdot [A + m' \cdot Av]$	(88)
LONG COLUMNS.	
$P = \dfrac{Ps}{1 + F \cdot K \cdot cf(l/d)^2}$ *(Gordon's Formula.)*	(89)
$F = Ac \cdot d^2/Ic$	(90)
$K = cu/(\pi^2 \cdot Ec)$	(91)
$P = \dfrac{Ps}{1 + K \cdot cf(l/g)^2}$ *(Rankine's Formula.)*	(92)

NUMERICAL EXAMPLES

MEMBERS UNDER CONCENTRIC LOADING

SHORT COLUMNS.

	Formula No.
DATA.—$A = 100$ sq. in., $Av = 4$ sq. in., $c = 600$ lb./in.^{2}, $m = 15$, $m' = 15$, $s = 15$, $V_t = 0\cdot01$	
(Vertical Reinforcement only.)	
$P = 600[100+14\times 4] = 93{,}600$ (lb.)	(81)
$p = [1 + 14 \times \cdot04]600 = 936$ (lb./in.^{2})	(83)
$r = (936 - 600)/(14 \times 600) = 0\cdot04$	(84)
$c = 936/(1 + 14 \times \cdot04) = 600$ (lb./in.^{2})	(85)
(Vertical and Transverse Reinforcement.)	
$P = (1+15\times\cdot01)[1+14\times\cdot04]600\times 100 = 107{,}640$ (lb.)	(86)
$p = (1+15\times\cdot01)[1+14\times\cdot04]600 = 1076\cdot4$ (lb./in.^{2})	(87)
(*French Commission Formula.*)	
$P = (1+15\times\cdot01)600[100+15\times 4] = 110{,}400$ (lb.)	(88)

LONG COLUMNS.

With Vertical and Transverse Reinforcement.

DATA.—$Ae = 156$ sq. in., $d = 10$ in., $Ec = 2000000$, $cf = 0\cdot25$, $Ie = 1729$ in.-units, $l/d = 30$, $l/g = 90$, $Ps = 107{,}640$ lb., $cu = 2400$ lb./in.^{2}

$F = (156 \times 10^2)/1729 = 9\cdot02$, say 10 . . (90)

$K = 2400/(9\cdot86 \times 2{,}000{,}000) = 0\cdot00012$ . . (91)

(*Gordon's Formula.*)

$$P = \frac{107{,}640}{1 + 10 \times \cdot00012 \times \cdot25 \times 30^2} = 84{,}755 \text{ (lb.)} \quad (89)$$

(*Rankine's Formula.*)

$$P = \frac{107{,}640}{1 + \cdot00012 \times \cdot25 \times 90^2} = 86{,}600 \text{ (lb.)}. \quad (92)$$

(II) COLUMNS UNDER ECCENTRIC LOADING

The following rules apply only to columns with the vertical reinforcing bars symmetrical in respect of the central axis, and where the eccentricity of the loading and the deflection of the column are not great enough to produce tension.

These conditions frequently obtain in ordinary reinforced concrete work and are assumed for the purpose of this section. Members to which they do not apply must be treated in accordance with the methods of calculation given in Chapter X for members subject to combined bending and direct stresses.

If the upper end of a column carrying an eccentric load is not fixed in position and direction, the maximum bending moment may be regarded as being at the base. But if the upper end is rigidly fixed, as in a building, the maximum bending moment will be at some other point, varying according to the conditions prevailing.

Let e = the *eccentricity* of the load, measured from the axis of the column when straight or denoted by the ratio $e = B/P$ Then the maximum bending moment is

$$B = P(e + dn) \qquad (93)$$

The deflection is, approximately

$$dn = \frac{B \cdot l^2}{2Ec \cdot Ie}$$

$$= \frac{P \cdot e \cdot l^2}{2Ec \cdot Ie} \qquad (94)$$

But, in accordance with the assumptions stated above, we may here neglect deflection and adopt the relation

$$B = P \cdot e \qquad (95)$$

The general equation for the algebraic sum of the stresses due to combined direct compressive and flexural stresses is

$$c + f = \frac{P}{A} + \frac{B}{M}$$

This equation may be employed in reinforced concrete design.

Then, with ac = *arm* of the *extreme* fibres of the concrete

measured from the axis of the column, Ie = inertia moment of the equivalent section, and Ae = the area of the equivalent section, the maximum and minimum compressive stresses in the concrete will be

$$c(\max) = \frac{P}{Ae} + \frac{B}{Ie/ae}$$

or, with M = modulus of section = Ie/ae

$$c(\max) = \frac{P}{Ae} + \frac{B}{M} \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$c(\min) = \frac{P}{Ae} - \frac{B}{Ie/ae}$$

or, substituting M, as above

$$c(\min) = \frac{P}{Ae} - \frac{B}{M} \quad . \quad . \quad . \quad . \quad . \quad (97)$$

The maximum compressive stress occurs in the fibres on the side nearest to the eccentric load, the minimum stress being at the opposite side. In accordance with the elastic theory, the corresponding compressive stresses in the steel, taking as = arm of the steel measured from the central axis, will be

$$cs(\max) = m \cdot \left(\frac{P}{Ae} + \frac{B}{Ie/ae}\right)$$

or, with Ms = modulus of section in terms of as

$$cs(\max) = m \cdot \left(\frac{P}{Ae} + \frac{B}{Ms}\right) \quad . \quad . \quad . \quad . \quad . \quad (98)$$

$$cs(\min) = m \cdot \left(\frac{P}{Ae} - \frac{B}{Ie/ae}\right)$$

or, substituting Ms, as above

$$cs(\min) = m \cdot \left(\frac{P}{Ae} - \frac{B}{Ms}\right) \quad . \quad . \quad (99)$$

Formulæ for the calculation of inertia moments will be found in Chapter XI.

Both short and long columns are treated in the same way, with the exception that the compressive stress in the case of a long column must be reduced proportionately as the ratio l/d or l/g is increased.

SUMMARY OF COMPRESSION MEMBER FORMULÆ

MEMBERS UNDER ECCENTRIC LOADING

Sections Symmetrical about Central Axis. Resultant Stresses wholly Compressive.

	Formula No.
$B = P(e + dn)$	(93)
$dn = \dfrac{P \cdot e \cdot l^2}{2Ec \cdot Ie}$	(94)
$B = P \cdot e$ (Deflection neglected)	(95)
$c_{(max)} = \dfrac{P}{Ae} + \dfrac{B}{Ie/ae}$	
$= \dfrac{P}{Ae} + \dfrac{B}{M}$	(96)
$c_{(min)} = \dfrac{P}{Ae} - \dfrac{B}{Ie/ae}$	
$= \dfrac{P}{Ae} - \dfrac{B}{M}$	(97)
$cs_{(max)} = m \cdot \left(\dfrac{P}{Ae} + \dfrac{B}{Ie/as}\right)$	
$= m \cdot \left(\dfrac{P}{Ae} + \dfrac{B}{Ms}\right)$	(98)
$cs_{(min)} = m \cdot \left(\dfrac{P}{A} - \dfrac{B}{Ie/as}\right)$	
$= m \cdot \left(\dfrac{P}{Ae} - \dfrac{B}{Ms}\right)$	(99)

NUMERICAL EXAMPLES

MEMBERS UNDER ECCENTRIC LOADING

Vertical and Transverse Reinforcement.

DATA.—$Ac = 156$ sq. in., $ac = 5$ in., $as = 4$ in., $Av = 4$ sq. in., $Ec = 2{,}000{,}000$, $e = 2$ in., $Ic = 1729$ in.-units, $l = 200$ in., $M = Ic/ac = 346$ in.-units, $Ms = Ic/as = 432$ in.-units, $P = 50{,}000$ lb.

	Formula No.
$B = 50{,}000\,(2 + \cdot 58) = 129{,}000 \text{ in.-lb.})$	(93)
$dn = \dfrac{50{,}000 \times 2 \times 200^2}{2 \times 2{,}000{,}000 \times 1729} = 0{\cdot}58 \text{ (in.)}$	(94)
$B_0 = 50{,}000 \times 2 = 100{,}000 \text{ (in.-lb.)}$	(95)
$c(\text{max}) = \dfrac{50{,}000}{156} + \dfrac{100{,}000}{346}$	
$= 320 + 289 = 609 \text{ (lb./in.}^2\text{)}$	(96)
$c(\text{min}) = 320 - 289 = 31 \text{ (lb./in.}^2\text{)}$	(97)
$cs(\text{max}) = 15\left(\dfrac{50{,}000}{156} + \dfrac{100{,}000}{432}\right)$	
$= 15(320 + 231) = 8265 \text{ (lb./in.}^2\text{)}$	(98)
$cs(\text{min}) = 15(320 - 231) = 1335 \text{ (lb./in.}^2\text{)}$	(99)

MEMORANDA

CHAPTER X

FORMULÆ FOR MEMBERS UNDER COMBINED STRESSES

Derivation of Formulæ.—In a member designed primarily for resistance to direct stress, the bending moment may be due either to the eccentricity of the principal force or to the application of a second force acting at a certain distance from the axis of the member. Conversely, in a member designed primarily for resistance to bending moment, the direct stress may be due either to the obliquity of the principal force or to the application of a second force acting parallel to the axis of the member.

As in reinforced concrete practice the direct stress is almost invariably compressive, it may be said that formulæ for combined stresses represent a combination of column and beam formulæ.

Leaving direct tension out of consideration, the resultant stress in a member subject to combined stresses may be (1) wholly compressive, or (2) compressive on one side of the axis and tensile on the other side of the axis. Thus, two cases are to be distinguished in dealing with members of this class.

Diagrams illustrating direct and bending stresses can be drawn with the centroidal axis of the member either vertical or horizontal so as to represent part of a column or part of a beam. As a matter of convenience, the diagrams in this chapter are drawn with the axis horizontal, the upper surface being designated the "loaded" side and the lower surface the "unloaded" side.

Case I.—In Fig. 85 let the direct compressive stress acting all over the section be represented by the area **ABCD**, where **AB** or **CD** is the stress intensity, and let the bending stresses be represented by the two areas **BOE** and **COF**, where **BE** is the compressive stress due to flexure and **FC** the tensile stress due to flexure.

Then the compressive fibre stress at the loaded side will be AB + BE = AE, and the compressive fibre stress at the unloaded side will be the difference between the compressive

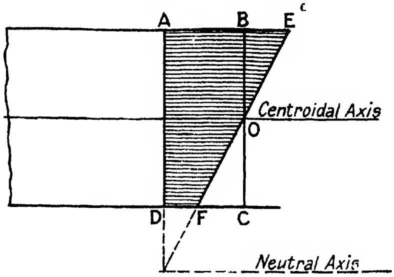


FIG. 85

and tensile stresses DC and FC, or DC − FC = DF. The distribution of the resultant fibre stresses is indicated in Fig. 85 by the lines in the area AEFD.

The diagram also shows that the neutral axis in this case is outside the member at the intersection of the lines AD and EF.

Case II.—If we now assume the bending moment to be increased so as to give the bending stress diagrams BOG and COH, as in Fig. 86, the maximum fibre stress at the

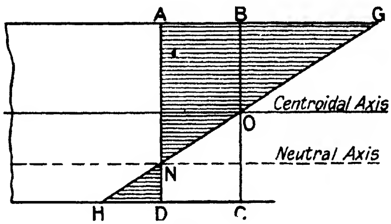


FIG. 86

loaded side becomes AB + BG = AG, and the direct compressive stress previously existing at the unloaded side is entirely neutralized by the tensile stress CH due to bending moment, and the difference between HC and CD represents the tensile fibre stress at the unloaded side, or HC − CD = HD.

The distribution of the resultant fibre stresses is indicated by the lines in the areas **ANG** and **DNH** above and below the neutral axis respectively.

It will be seen that, under the conditions assumed, the resultant fibre stress at the loaded side is always compressive, while that at the unloaded side may be either compressive or tensile. But in some members subject to combined stresses, there may be a complete reversal of the stresses, involving a corresponding change in the nature of the stresses at the loaded and unloaded sides, respectively.

Therefore, the use of the conventional symbols c = compressive stress and t = tensile stress would lead to confusion, and to obviate anything of the kind distinctive symbols are employed for the various stresses, as in the subjoined table and in Figs. 87 and 88.

NOTATION FOR HOMOGENEOUS MATERIALS

A = *Area* of the section = $b{\cdot}D$

ae = *arm* of the *extreme* fibres above or below the centroidal axis = $\frac{1}{2}D$

B = *Bending* moment at the section = $P{\cdot}e$

b = *breadth* of the section

D = *Depth* of the section

e = *eccentricity* of R, or the distance from the point where R intersects the section to the centroidal axis.

f = *flexural* stress intensity (either compressive or tensile) at the extreme fibres.

I = *Inertia* moment = $(\frac{1}{12}b{\cdot}D^3) = (\frac{1}{3}A{\cdot}ae^2)$

M = *Modulus* of section = I/ae

P = Component of R normal to the section = Total *push* or *pull*

p = Direct stress intensity (*push* or *pull*) = $P/b{\cdot}D = P/A$

R = *Resultant* of the external forces acting on the section.

Σs = *Sum* of the extreme fibre *stress* intensities = $(p + f)$ or = $(p - f)$

A member is said to be subject to combined stresses if the resultant R of the external forces acting on a given cross section passes through the section, or through an imaginary

continuation of the section, at any point above or below the centroidal axis.

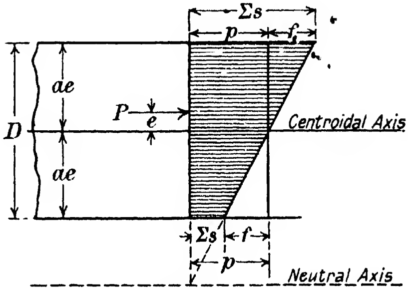


FIG. 87

The component of the resultant, normal to the section, can be replaced by an axial force (push or pull) P, and a bending moment B, the arm of leverage of which is represented by e the eccentricity of R, or the distance between

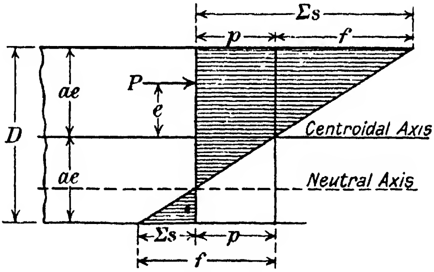


FIG. 88

the centroidal axis and the point where the section is intersected by R.

Then we have for the direct stress per unit of area

$$p = \frac{P}{b \cdot D} = \frac{P}{A}$$

and for the flexural stress at the extreme fibres

$$f = \frac{B}{I/ae} = \frac{P \cdot e}{I/ae}$$

Consequently, as represented in Figs. 87 and 88, we obtain for the sum of the extreme fibre stresses

At the Loaded Side

$$\Sigma s = (p + f) = \frac{P}{A} + \frac{P \cdot e}{I/ae}$$

$$= P\left(\frac{1}{A} + \frac{e}{I/ae}\right) \quad . \quad . \quad . \quad (100)$$

At the Unloaded Side

$$\Sigma s = (p - f) = \frac{P}{A} - \frac{P \cdot e}{I/ae}$$

$$= P\left(\frac{1}{A} - \frac{e}{I/ae}\right) \quad . \quad . \quad . \quad (101)$$

In order that the stress at the unloaded side may be of the same sign as the direct stress, as in Fig. 87, the quantity $\frac{1}{A}$ must be greater than $\frac{e}{I/ae}$. Where the two are of equal value, the stress will be zero at the unloaded side, and of twice the intensity of the direct stress at the loaded side; in such an event, $\frac{1}{A} = \frac{e}{I/ae}$ and $\frac{I/ae}{A} = e$. Therefore, in order to obviate any change of sign or reversal of stress, as in Fig. 88, the eccentricity or line of action of the external forces must not be at a distance greater than $\frac{I/ae}{A}$ from the axis of the section. As $I = \frac{1}{12} b \cdot D^3$, $ac = \frac{1}{2} D$ and $A = b \cdot D$, we obtain for the limiting distance—

$$\frac{I/ae}{A} = \frac{\frac{1}{12} b \cdot D^3 / \frac{1}{2} D}{b \cdot D}$$

$$= \tfrac{1}{6} D = 0{\cdot}1\dot{6}\, D$$

Equations (100) and (101) apply only to rectangular members of homogeneous materials, and are given with the object of making clear the derivation of the formulæ which follow for reinforced concrete members subject to combined stresses, and of affording some guidance in the application

of the formulæ, which, in order to avoid unnecessarily complicated equations, are confined to those for rectangular sections.

It should be noted that (100) and (101) are essentially similar to equations (96) and (97) given on page 199 for columns under eccentric loading. The relationship can be made clear at a glance by substituting in (100) and (101), M = modulus of section in terms of ae, or $M = I/ae$.

Thus, (100) could be written—

$$\Sigma s = P\left(\frac{1}{A} + \frac{e}{M}\right)$$
$$= \frac{P}{A} + \frac{P \cdot e}{M}$$
$$= \frac{P}{A} + \frac{B}{M}$$

and (101) would become

$$\Sigma s = P\left(\frac{1}{A} - \frac{e}{M}\right)$$
$$\frac{P}{A} - \frac{P \cdot e}{M}$$
$$\frac{P}{A} - \frac{B}{M}$$

COMPRESSION AND BENDING

The following equations are of general applicability to reinforced concrete members subject to combined stresses, whether the direct stress is tensile or compressive, as explained at the end of this chapter.

In the case of reinforced concrete, it is necessary to distinguish between the stresses in the concrete and the steel, and to make other distinctions which are not required in connection with homogeneous materials. Therefore additional symbols must be employed as in the subjoined table, and in Figs. 89 to 95.

NOTATION FOR REINFORCED CONCRETE

A = *Area* of the section $= b \cdot D$

Ac = *Equivalent area* of the section

As = Area of the steel below centroidal axis, or above and below central axis in symmetrical sections

As' = Area of the steel above centroidal axis

ae = *arm* of the *extreme* fibres below centroidal axis, or above and below central axis in symmetrical sections

ae' = *arm* of the *extreme* fibres above centroidal axis

as = *arm* of the *steel* below centroidal axis, or above and below central axis in symmetrical sections

as' = *arm* of the *steel* above centroidal axis

B = *Bending* moment at the section $= P \cdot e$

b = *breadth* of the section

D = *Depth* of the section

e = *eccentricity* of R

Ie = *Inertia* moment of the *equivalent* area $= (Ic + Is)$

i = *inset* of the steel from unloaded surface, or from loaded and unloaded surfaces in symmetrical sections

i' = inset of the steel from loaded surface

P = Component of R, normal to section. (*Push* or *Pull*)

p = Direct stress intensity (*push* or *pull*) $= P/b \cdot D = P/A$

R = *Resultant* of the external forces acting on the section

rc = *resultant* stress at the extreme fibres of the *concrete* below centroidal axis

rc' = *resultant* stress at the extreme fibres of the *concrete* above centroidal axis

rs = *resultant* stress in the *steel* below centroidal axis

rs' = *resultant* stress in the *steel* above centroidal axis

Formulæ Based upon General Equation for Combined Stresses.—From (100) and (101), modified in accordance with Figs. 89 and 90, we have for the resultant stress, or the sum of the extreme fibre stresses, in the concrete

At the Loaded Side—

$$rc' = \frac{P}{Ae} + \frac{B}{Ie/ae'}$$

$$= P\left(\frac{1}{Ae} + \frac{e}{Ie/ae'}\right) \quad . \quad . \quad . \quad (102)$$

At the Unloaded Side—

$$rc = \frac{P}{Ae} - \frac{B}{Ie/ae}$$

$$= P\left(\frac{1}{Ae} - \frac{e}{Ie/ae}\right) \quad . \quad . \quad . \quad (103)$$

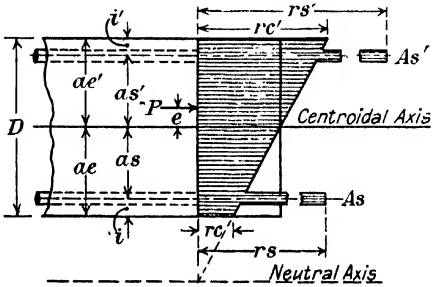


FIG. 89

Similarly, for the resultant stress in the steel, we have

At the Loaded Side—

$$rs' = m \cdot \frac{P}{Ae} + m \cdot \frac{B}{Ie/as'}$$

$$= m \cdot P\left(\frac{1}{Ae} + \frac{e}{Ie/as'}\right) \quad . \quad . \quad (104)$$

At the Unloaded Side—

$$rs = m \cdot \frac{P}{Ae} - m \cdot \frac{B}{Ie/as}$$

$$= m \cdot P\left(\frac{1}{Ae} - \frac{e}{Ie/as}\right) \quad . \quad . \quad (105)$$

For approximate purposes the values of the resultant stresses in the steel may be taken at

$$rs' = m \cdot rc'$$

$$rs = m \cdot rc$$

Reference to Figs. 89 and 90 will show that the values given by these approximate equations must always be somewhat above the true values for rs' and somewhat below the true values for rs.

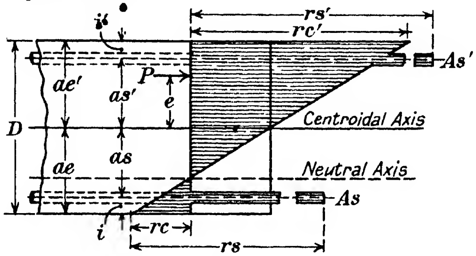


FIG. 90

The distance of the centroidal axis, passing through the centroid of the equivalent area Ac, is given by the equation

$$ae' = \frac{\frac{1}{2}b \cdot D^2 + m[As' \cdot i' + As(D - i)]}{b \cdot D + m(As' + As)} \quad . \quad . \quad (106)$$

Having found the position of the centroidal axis, the resultant stresses in the concrete and the steel can be calculated by formulæ (102) to (105).

Simplified Formulæ for Symmetrical Reinforcement.— In members where the reinforcement is arranged symmetrically above and below the centroidal axis, Equations (102) to (106) assume simplified forms owing to the smaller number of symbols required.

Thus, as $As = As'$, $i = i'$, $ae = ae'$ and $as = as'$ (see Figs. 91 and 92).

Equation (106) reduces to

$$ae = \frac{\frac{1}{2}b \cdot D^2 + m \cdot As[i + (D - i)]}{b \cdot D + 2m \cdot As}$$

$$= \frac{\frac{1}{2}b \cdot D^2 + m \cdot As \cdot D}{b \cdot D + 2m \cdot As}$$

$$= \frac{1}{2}D \quad . \quad . \quad . \quad . \quad . \quad (106a)$$

Equations (102) and (103) become

$$rc' = P\left(\frac{1}{Ae} + \frac{e}{Ie/ae}\right) \quad . \quad . \quad . \quad . \quad (102a)$$

$$rc = P\left(\frac{1}{Ae} - \frac{e}{Ie/ae}\right) \quad . \quad . \quad . \quad . \quad (103a)$$

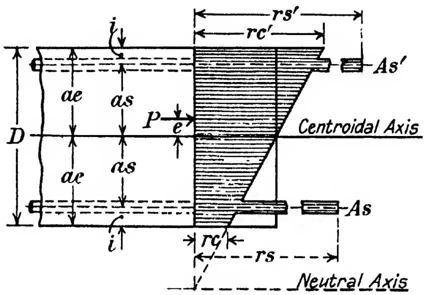


FIG. 91

Equations (104) and (105) become

$$rs' = m{\cdot}P\left(\frac{1}{Ae} + \frac{e}{Ie/as}\right) \quad . \quad . \quad . \quad . \quad (104a)$$

$$rs = m{\cdot}P\left(\frac{1}{Ae} - \frac{e}{Ie/as}\right) \quad . \quad . \quad . \quad . \quad (105a)$$

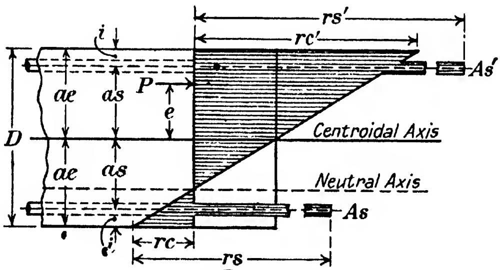


FIG. 92

Note on Formulæ (102) to (106a.)—When using the foregoing equations, it is necessary to ascertain whether the member under consideration comes under Case I or Case II. This point can easily be settled by comparison of the

values calculated for $1/Ae$ and $e/(Ie/ae')$, or for $1/Ae$ and $e/(Ie/ae)$.

Thus, in order that the resultant stress at the unloaded side of a section shall not be of opposite sign to the direct stress, or to the resultant stress at the loaded side, the quantity $1/Ae$ must be greater than, or equal to, $e/(Ie/ae')$.

If the direct stress is compressive there will be compressive stress throughout the section providing $1/Ae$ is greater than $e/(Ie/ae')$, and if the two quantities are of equal value the compressive stress will diminish to zero at the unloaded side.

Again, if $1/Ae$ is less than $e/(Ie/ae')$ there will be compressive stress at the loaded side and tensile stress at the unloaded side.

Therefore, to obviate any reversal of stress in the section, e must not be greater than $(Ie/ae')/Ae$.

We have already shown that in homogeneous members of rectangular section, there will be a reversal of stress at the unloaded side if the quantity e is greater than $\frac{1}{6}D$.

The limit is higher for reinforced concrete members, the actual value varying with the proportion of reinforcement employed, as shown graphically in Fig. 93, which has been calculated for ratios of symmetrical reinforcement from 0 to 0·1, on the basis $r = (As + As')b{\cdot}D$.

In members where tension is developed the tensile resistance of the concrete should be neglected, as in reinforced concrete beam design.

The relationship between formulæ (102) to (105*a*) and (96) and (97) may be illustrated by employing symbols denoting the section modulus in terms of ac', ac, as' and as.

Thus, we could write:

(102) with $M' = Ie/ae'$

$$rc' = \frac{P}{Ae} + \frac{B}{M'} \qquad = P\left(\frac{1}{Ae} + \frac{e}{M'}\right).$$

(103) with $M = Ie/ae$

$$rc = \frac{P}{Ae} - \frac{B}{M} \qquad = P\left(\frac{1}{Ae} - \frac{e}{M}\right)$$

(104) with $Ms' = Ie/as'$

$$rs' = m\left(\frac{P}{Ae} + \frac{B}{Ms'}\right) = m\cdot P\left(\frac{1}{Ae} + \frac{e}{Ms'}\right)$$

(105) with $Ms = Ie/as$

$$rs = m\left(\frac{P}{Ae} - \frac{B}{Ms}\right) = m\cdot P\left(\frac{1}{Ae} - \frac{e}{Ms}\right)$$

Formulæ (102*a*) to (105*a*) could be treated in the same way. The additional symbols involved in these variations are scarcely necessary, except perhaps as convenient abbreviations

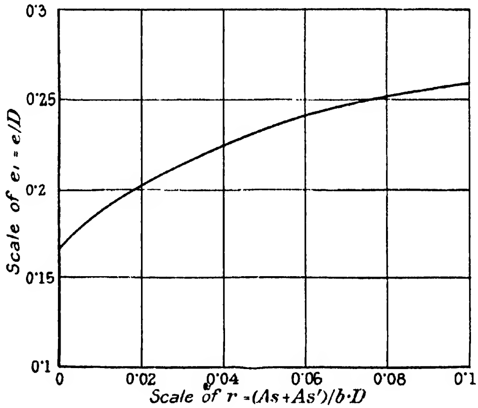


FIG. 93

Formulæ Based upon the Theory of Simple Flexure.— Members subject to combined stresses and coming under Case II, where the eccentricity of R is such that tension is developed on one side of the section, can most conveniently be calculated by a method similar to that adopted for reinforced concrete beams. So far as concerns combined direct compressive and flexural stresses, the resulting equations may be said to constitute a combination of pillar and beam formulæ, but it should be noted that they are also available for the calculation of members under direct tensile and flexural stresses.

The equations differ in form from reinforced concrete beam equations, because in addition to the neutral axis n there is a central axis about which the bending moment B is taken, and because of the factors necessitated by the existence of the external force P.

Consequently, besides assuming distinctive forms, the equations embody factors and symbols which do not appear in beam formulæ.

Nevertheless, as shown later, several of the equations can be expressed in terms practically identical with those of beam formulæ. It should be noted that by making $P = O$, the formulæ for combined direct and flexural stresses give precisely the same results as those for simple flexure, and conversely that by making $B = O$ the formulæ are precisely equivalent to those for direct compression.

As the equivalent area Ae is not employed, the centroidal axis, required in the case of the preceding series of equations, is replaced by a central axis, as shown in Fig. 94, the neutral axis being at a variable distance n from the " loaded " side of the member.

Then for P the component of the resultant total force normal to the cross section, and for B the bending moment about the central axis,

We have

$$P = \tfrac{1}{2}rc'\cdot b\cdot n + rs'\cdot As' - rs\cdot As \quad . \quad . \quad . \quad (107)$$

and

$$B = \tfrac{1}{2}rc'\cdot b\cdot n(\tfrac{1}{2}D - \tfrac{1}{3}n) + rs'\cdot As'\cdot as' + rs\cdot As\cdot as \quad (108)$$

In accordance with the hypothesis of the conservation of plane sections, the resultant fibre stresses in the steel above and below the neutral axis can be computed by equations similar to those for the stresses in the reinforcement of rectangular beams.

Thus, referring to Fig. 94, we see that for the resultant fibre stress in the steel above the neutral axis we have

$$rs' = \frac{as' - \tfrac{1}{2}D + n}{n}\cdot m\cdot rc'$$

In case this should not be clear to the reader it may be pointed out that

$$as' - \tfrac{1}{2}D + n = as' + n - \tfrac{1}{2}D$$
$$= n - (\tfrac{1}{2}D - as')$$

But as by Fig. 94,

$$(\tfrac{1}{2}D - as') = i'$$

we obtain

$$n - (\tfrac{1}{2}D - as') = n - i'$$

and the formula can be written in terms corresponding with those of the equivalent beam formula, or

$$rs' = \frac{(n - i')}{n} \cdot m \cdot rc' \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

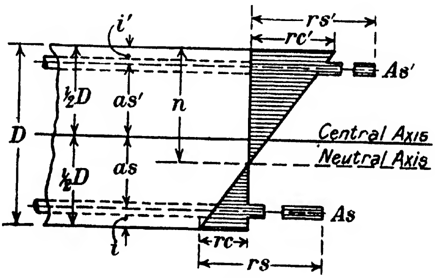


FIG. 94

Similarly, for the resultant fibre stress in the steel below the neutral axis,

We have

$$rs = \frac{as + \tfrac{1}{2}D - n}{n} \cdot m \cdot rc'$$

By reference to Fig. 94 it will be seen that

$$as + \tfrac{1}{2}D - n = (D - i) - n$$

Here $(D - i)$ corresponds with d = effective depth, as employed in beams. Therefore the equation could be written in terms identical with those of a beam formula, or

$$rs = \frac{(d-n)}{n} \cdot m \cdot rc'$$

If this equation were so written, however, the introduction of the symbol d would lead to confusion in the present series of formulæ where D is the basis of measurements relating to depth.

Consequently a preferable equivalent is

$$rs = \frac{(D-i)-n}{n}\cdot m\cdot rc'$$

$$= \frac{D-(i+n)}{n}\cdot m\cdot rc' \quad . \quad . \quad . \quad . \quad (110)$$

The combination of formulæ (107) to (110) results in an equation for the determination of n, the depth of the neutral axis, as follows—

$$\begin{aligned} n^3\cdot\tfrac{1}{6}P - n^2(\tfrac{1}{4}P\cdot D - \tfrac{1}{2}B) + n\cdot m/b\,[B(As' + As) - P(As'\cdot as') \\ - As\cdot as] - B\cdot m/b[As'\cdot i' + As\,(D-i)] \\ - P\cdot m/b[As\cdot as(D-i) - As'\cdot as'\cdot i'] = 0 \quad . \quad (111) \end{aligned}$$

From this cubic equation, which can most conveniently be solved by successive approximations, is obtained the following formula for the resultant extreme fibre stress in the concrete above the neutral axis

$$rc' = \frac{P\cdot n}{\tfrac{1}{2}b\cdot n^2 + m\cdot As'(n-i') - m\cdot As[D-(i+n)]} \quad (112)$$

Finally, for the resultant extreme fibre stress in the concrete below the neutral axis, we have, as shown by the stress diagram (Fig. 94)—

$$rc = \frac{(D-n)}{n}\cdot rc'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

Simplified Formulæ for Symmetrical Reinforcement.—If the reinforcement is arranged symmetrically about the central axis, we have $As = As'$ and $as = as'$ (compare Figs. 94 and 95), and can adopt the following series of simplified formulæ—

$$P = \tfrac{1}{2}rc'\cdot b\cdot n + (rs' - rs)\,As \qquad . \quad (107a)$$

$$B = \tfrac{1}{2}rc'\cdot b\cdot n(\tfrac{1}{2}D - \tfrac{1}{3}n) + (rs' + rs)As\cdot as \qquad . \quad (108a)$$

$$rs' = \frac{as - \frac{1}{2}D + n}{n}\cdot m\cdot rc'$$

$$= \frac{(n-i)}{n}\cdot m\cdot rc' \qquad . \quad . \quad . \quad . \quad . \quad (109a)$$

$$rs = \frac{as + \frac{1}{2}D - n}{n}\cdot m\cdot rc'$$

$$= \frac{D - (i+n)}{n}\cdot m\cdot rc' \qquad . \quad . \quad . \quad . \quad (110a)$$

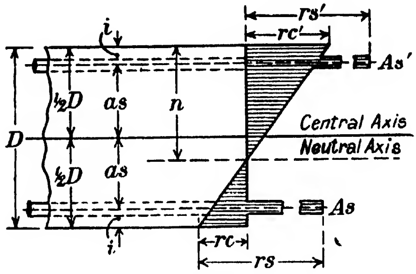


FIG. 95

Similarly, the cubic Equation (111) reduces to

$$n^3\cdot\tfrac{1}{6}P - n^2(\tfrac{1}{4}P\cdot D - \tfrac{1}{2}B) + 2n\cdot B\cdot m\cdot As/b - m\cdot As/b$$
$$(B\cdot D + 2P\cdot as^2) = 0 \qquad . \quad . \quad . \quad . \quad (111a)$$

and (112) becomes

$$rc' = \frac{P}{\frac{1}{2}b\cdot n + m\cdot As\left(\dfrac{2n - D}{n}\right)} \qquad . \quad . \quad . \quad (112a)$$

$$rc = \frac{(D-n)}{n}\cdot rc' \qquad . \quad . \quad . \quad . \quad . \quad (113a)$$

The application of Formula (111*a*) for symmetrical reinforcement is greatly facilitated by the aid of a diagram such as that reproduced as Fig. 96.

When the position of the neutral axis has been determined, the values of the resultant stresses can be obtained by Equations (109a), (110a), (112a) and (113), which, together with (107a) and (108a), can be further simplified by the use of the ratios: $as_{,} = as/D$; $i_{,} = i/D$; $n_{,} = n/D$; $r = As/b{\cdot}D$.

Thus

$$P = [\tfrac{1}{2}rc'{\cdot}n_{,} + r(rs' - rs)]b{\cdot}D \quad . \quad . \quad . \quad (107b)$$

$$B = [\tfrac{1}{2}rc'{\cdot}n_{,}(\tfrac{1}{2} - \tfrac{1}{3}n_{,}) + r(rs' + rs)as_{,}]b{\cdot}D^2 \quad . \quad (108b)$$

$$rs' = \frac{n_{,} - i_{,}}{n_{,}}{\cdot}m{\cdot}rc' \quad . \quad . \quad . \quad . \quad . \quad (109b)$$

$$rs = \frac{1 - (i_{,} + n_{,})}{n_{,}}{\cdot}m{\cdot}rc' \quad . \quad . \quad . \quad . \quad (110b)$$

$$rc' = \frac{P}{\left[\tfrac{1}{2}n_{,} + m{\cdot}r\left(\frac{2n_{,} - 1}{n_{,}}\right)\right]b{\cdot}D} \quad . \quad . \quad . \quad (112b)$$

$$rc = \frac{1 - n_{,}}{n_{,}}{\cdot}rc' \quad . \quad . \quad . \quad . \quad . \quad (113b)$$

Referring to Equation (111a) it will be noted that, with given dimensions, n is governed solely by the ratio $B/P = e$.

Therefore B/P in (111a) can be expressed as a function of n, and thereby we can obtain

$$\frac{B}{P} = e = \frac{- n^3 + 3n^2{\cdot}\tfrac{1}{2}D + 12m{\cdot}As{\cdot}as^2/b}{3n^2 + 12n{\cdot}m{\cdot}As/b - 6D{\cdot}m{\cdot}As/b} \quad . \quad . \quad (114)$$

Adopting the ratios $n_{,} = n/D$; $r = As/b{\cdot}D$ and $as_{,} = as/D$, Equation (114) reduces to

$$\frac{B}{P} = e = \left[\frac{- n_{,}^3 + 1\tfrac{1}{2}n_{,}^2 + 12m{\cdot}r{\cdot}as_{,}^2}{3n_{,}^2 + 12n_{,}{\cdot}m{\cdot}r - 6m{\cdot}r}\right]D \quad . \quad . \quad (114a)$$

Then, dividing both sides of (114a) by D and taking $m = 15$ and $as_{,} = 0{\cdot}4$

we have

$$\frac{B}{P{\cdot}D} = e_{,} = \frac{- n_{,}^3 + 1\tfrac{1}{2}n_{,}^2 + 28{\cdot}8r}{3n_{,}^2 + 180n_{,}{\cdot}r - 90r} \quad (115)$$

If preferred, the value of $as_{\prime}$ can be taken at $0 \cdot 408$ so as to give the simple quantity $30r = (12 \times 15 \times 0 \cdot 408^2)r$, instead of $28 \cdot 8r$ in the numerator. The advantage so gained in respect of simplicity is counterbalanced by the complication introduced into the value of $i_{\prime}$, which becomes $0 \cdot 5 - 0 \cdot 408 = 0 \cdot 092$.

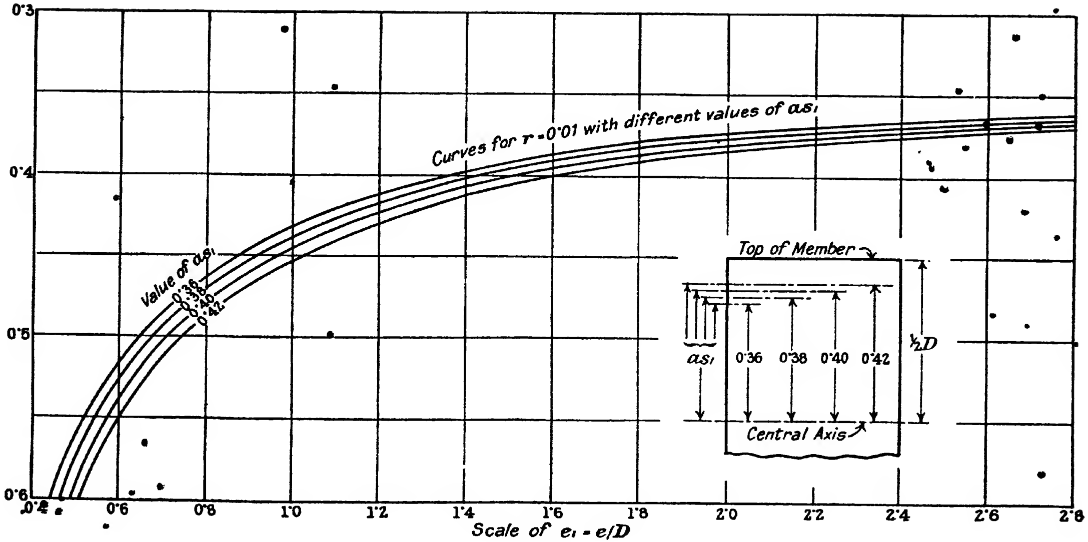
Similarly, the value $as_{\prime} = 0 \cdot 42$, as recommended in some text-books leads to the quantity $31 \cdot 75r$ in the numerator of (115) and to the value $i_{\prime} = 0 \cdot 08$.

On the whole, the value $as_{\prime} = 0 \cdot 4$ is the most convenient for general use, especially as it leads to the simple value $i_{\prime} = 0 \cdot 1$.

For assumed ratios of reinforcement, r, values of $e_{\prime}$ can be computed by (115) for different values of $n_{\prime}$, and by plotting the results as in Fig. 96 a series of curves is obtained by the aid of which, together with known values of B and P, the proportionate depth of the neutral axis can be readily found for use in formulæ (107*a*) to (113*a*), and (107*b*) to (112*b*). Intermediate values sufficiently accurate for all practical purposes can be obtained by interpolation.

Fig. 96 is based upon the value $as = 0 \cdot 4D$, or $as_{\prime} = 0 \cdot 4$. In the case of members where it is desired to arrange the reinforcement so as to make $as_{\prime}$ either greater or less than $0 \cdot 4$, the diagram can still be used with approximately accurate results by taking values of $n_{\prime}$ slightly below or above those given by the curves. As shown by Fig. 97 the differences in the values of $n_{\prime}$ for variations of $as_{\prime}$ from $0 \cdot 36$ to $0 \cdot 42$ are comparatively small, and for small variations they scarcely exceed the fractional errors usually permissible in practical computations.

Owing to the form in which equation (115) is expressed, the value of As to be inserted is either that in the area below the central axis or that in the area above the central axis, and the curves in Fig. 96 denote ratios of steel for the same area, or $r = As/b \cdot D$. Therefore in the case of a member where $r = 0 \cdot 01$ below the axis and $r = 0 \cdot 01$ above the axis, we must use the $r = 0 \cdot 01$ curve, and not the $r = 0 \cdot 02$ curve.



0·3
0·4
0·5
0·6
0·4
0·6
0·8
1·0
1·2
1·4
1·6
1·8
2·0
2·2
2·4
2·6
2·8
Curves for r = 0·01 with different values of as₁
Value of as₁
0·36
0·38
0·40
0·42
Top of Member
Central Axis
½D
as₁
Scale of e₁ = e/D

FIG. 97

As this method of procedure has been adopted hitherto in connection with similar diagrams in other treatises, the author has followed the general practice, although thinking it would be preferable for the inscriptions on the curves to denote total ratios of steel, such as $(r + r) = (As + As)/b{\cdot}D$, or $(r + r') = (As + As')/b{\cdot}D$.

So far as the employment of the diagram is concerned, however, it does not matter which method of designating the curves is adopted, so long as the user is clear as to the basis of the ratios.

The following examples illustrate the use of the diagram in finding values of $n_{\prime}$ for given values of $e_{\prime}$ and r.

EXAMPLE 1.—*Take a beam* 9 *in. in breadth,* 20 *in. in depth, reinforced with two* $\frac{3}{4}$ *in. diameter bars inset* 2 *in. below the upper surface, and with two* $\frac{3}{4}$ *in. diameter bars inset* 2 *in. above the lower surface. Let* $P = 30{,}000$ *lb,. and* $B = 250{,}000$ *in.-lbs. Find the depth* (n) *of the neutral axis.*

The ratio of steel to be taken into account is

$$r = As/(b{\cdot}D) = 2 \times 0{\cdot}44166/(9 \times 20) = 0{\cdot}005$$

The eccentricity ratio of the resultant of the external forces is

$$e_{\prime} = B/P{\cdot}D = 250{,}000/(30{,}000 \times 20) = 0{\cdot}4166$$

Turning to Fig. 96, we find that the intersection of $e_{\prime} = 0{\cdot}4166$ on the top scale and the curve $r = 0{\cdot}005$ is opposite $n_{\prime} = 0{\cdot}56$ on the left-hand scale.

Hence the neutral axis depth is

$$n = n_{\prime}{\cdot}D = 0{\cdot}56 \times 20 = 11{\cdot}2 \text{ (inches)}$$

EXAMPLE 2.—*Take a beam where* $b = 10$ *in.,* $D = 20$ *in.,* $As = 1{\cdot}6$ *sq. in., below the central axis (with* $As = 1{\cdot}6$ *sq. in. above the central axis),* $e_{\prime} = 0{\cdot}4$. *Find the depth of the neutral axis.*

In the first place, we calculate the value of $r = As/(b{\cdot}D) = 1{\cdot}6/(10 \times 20) = 0{\cdot}008$. Then, using the enlarged diagram at the bottom of Fig. 96, we find the intersection of $e_{\prime} = 0{\cdot}4$ and $r = 0{\cdot}008$ is opposite $n_{\prime} = 0{\cdot}65$ on the right-hand side. Consequently,

$$n = n_{\prime}{\cdot}D = 0{\cdot}65 \times 20 = 13 \text{ (inches)}$$

Summary of Formulæ and Numerical Examples.— At the end of this chapter we append a series of tables giving formulæ (102) to (115) in four groups for convenient reference, opposite each table being numerical examples of the equations. Two tables of numerical examples are also given showing the manner in which curves can be calculated for diagrams such as Fig. 96, and two tables of calculations illustrating the method to be adopted in the indirect solution of the cubic equations (111) and (111*a*) for the neutral axis depth.

A series of Standard Formulæ will be found on the folding plate facing p. 202.

TENSION AND BENDING

From the statement of general principles at the commencement of this section, it is evident that the effects produced by the combination of tension with bending moment are equivalent to those of combined compression and bending moment, the only difference being that the introduction of a negative force $-P$, in place of the positive force P, involves a corresponding change in the signs of the resultant stresses.

Consequently, the formulæ already given for compression and bending moment can be applied to the treatment of members subject to tension and bending moment combined.

SUMMARY OF FORMULÆ FOR MEMBERS UNDER COMBINED STRESSES

MEMBERS WHERE THE RESULTANT STRESSES ARE WHOLLY COMPRESSIVE

Formulæ Based upon the General Equation for Combined Stresses in Members of Homogeneous Materials.

$$rc' = P\left(\frac{1}{Ae} + \frac{e}{Ie/ae'}\right) \quad . \quad . \quad . \quad . \quad . \quad (102)$$

$$rc = P\left(\frac{1}{Ae} - \frac{e}{Ie/ae}\right) \quad . \quad . \quad . \quad . \quad . \quad (103)$$

$$rs' = m \cdot P\left(\frac{1}{Ae} + \frac{e}{Ie/as'}\right) \quad . \quad . \quad . \quad . \quad . \quad (104)$$

$$rs = m \cdot P\left(\frac{1}{Ae} - \frac{e}{Ie/as}\right) \quad . \quad . \quad . \quad . \quad . \quad (105)$$

$$ae' = \frac{\frac{1}{2}b \cdot D^2 + m[As' \cdot i' + As(D - i)]}{b \cdot D + m(As' + As)} \quad . \quad . \quad . \quad (106)$$

(*Simplified Formulæ for Symmetrical Sections.*)

$$rc' = P\left(\frac{1}{Ae} + \frac{e}{Ie/ae}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102a)$$

$$rc = P\left(\frac{1}{Ae} - \frac{e}{Ie/ae}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103a)$$

$$rs' = m \cdot P\left(\frac{1}{Ae} + \frac{e}{Ie/as}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (104a)$$

$$rs = m \cdot P\left(\frac{1}{Ae} - \frac{e}{Ie/as}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (105a)$$

$$ae = \tfrac{1}{2}D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106a)$$

NUMERICAL EXAMPLES

MEMBERS WHERE THE RESULTANT STRESSES ARE WHOLLY COMPRESSIVE.

DATA.—$Ae = 242$ sq. in., $ae = 10{\cdot}5$ in., $ae' = 9{\cdot}5$ in., $As = 1$ sq. in., $As' = 2$ sq. in., $as = 9{\cdot}5$ in., $as' = 7{\cdot}5$ in., $b = 10$ in., $D = 20$ in., $e = 2$ in., $Ie = 9554$ in. units, $i = 1$in., $i' = 2$ in., $m = 15$, $P = 50{,}000$ lb.

$$ae' = \frac{\frac{1}{2} \times 10 \times 20^2 + 15[2 \times 2 + 1(20 - 1)]}{10 \times 20 + 15(2 + 1)} = 9{\cdot}5 \text{ (in.)} \qquad (106)$$

$$rc' = 50{,}000(1/242 + 2/1000) = 306 \text{ (lb./in.}^2) \quad . \quad . \qquad (102)$$

$$rc = 50{,}000(1/242 - 2/910) = 96 \text{ (lb./in.}^2) \quad . \quad . \qquad (103)$$

$$rs' = 15 \times 50{,}000(1/242 + 2/1274) = 4{,}275 \text{ (lb./in.}^2) \qquad (104)$$

$$rs = 15 \times 50{,}000(1/242 - 2/1000) = 1{,}599 \text{ lb./in.}^2) \quad . \qquad (105)$$

(Symmetrical Sections.)

DATA.—As above, with the exception of the following: $As = 1{\cdot}5$ sq. in. above axis and $1{\cdot}5$ sq. in. below axis, $ae = 10$ in., $as = 8$ in., $Ie = 9354$ in. units, $i = 2$ in.

$$ae = \tfrac{1}{2} \times 20 = 10 \text{ (in.)} \quad . \quad . \quad . \quad . \quad . \qquad (106a)$$

$$rc' = 50{,}000(1/242 + 2/935) = 314 \text{ (lb./in.}^2) \quad . \quad . \qquad (102a)$$

$$rc = 50{,}000(1/242 - 2/935) = 100 \text{ (lb./in.}^2) \quad . \quad . \qquad (103a)$$

$$rs' = 15 + 50{,}000(1/242 + 2/1170) = 4{,}374 \text{ (lb./in.}^2) \qquad (104a)$$

$$rs = 15 \times 50{,}000(1/242 - 2/1170) = 1{,}824 \text{ (lb./in.}^2) \qquad (105a)$$

SUMMARY OF FORMULÆ FOR MEMBERS UNDER COMBINED STRESSES

MEMBERS WHERE THE RESULTANT STRESSES ARE COMPRESSIVE ON ONE SIDE OF THE AXIS AND TENSILE ON THE OTHER SIDE OF THE AXIS

Formulæ Based upon the Theory of Simple Flexure.

$$P = \tfrac{1}{2}rc'\cdot b\cdot n + rs'\cdot As' - rs\cdot As \qquad (107)$$

$$B = \tfrac{1}{2}rc'\cdot b\cdot n(\tfrac{1}{2}D - \tfrac{1}{3}n) + rs'\cdot As'\cdot as' + rs\cdot As\cdot as \qquad (108)$$

$$rs' = \frac{as' - \tfrac{1}{2}D + n}{n}\cdot m\cdot rc'$$

$$= \frac{(n - i')}{n}\cdot m\cdot rc' \qquad (109)$$

$$rs = \frac{as + \tfrac{1}{2}D - n}{n}\cdot m\cdot rc'$$

$$= \frac{D - (i + n)}{n}\cdot m\cdot rc' \qquad (110)$$

$$0 = n^3\cdot\tfrac{1}{6}P - n^2(\tfrac{1}{4}P\cdot D - \tfrac{1}{2}B)$$

$$+ n\cdot m/b[B(As' + As) - P(As'\cdot as' - As\cdot as)]$$

$$- B\cdot m/b[As'\cdot i' + As(D - i)]$$

$$- P\cdot m/b\,[As\cdot as(D - i) - As'\cdot as'\cdot i'] \qquad (111)$$

$$rc' = \frac{P\cdot n}{\tfrac{1}{2}b\cdot n^2 + m\cdot As'(n - i') - m\cdot As[D - (i + n)]} \qquad (112)$$

$$rc = \frac{D - n}{n}\cdot rc' \qquad (113)$$

NUMERICAL EXAMPLES

MEMBERS WHERE THE RESULTANT STRESSES ARE COMPRESSIVE ON ONE SIDE OF THE AXIS AND TENSILE ON THE OTHER SIDE OF THE AXIS

(*Unsymmetrical Section.*)

DATA.—$As = 1$ sq. in., $As' = 1{\cdot}2$ sq. in., $as = 8$ in., $as' = 7$ in., $b = 10$ in., $D = 20$ in., $c = 8{\cdot}3$ in., $i' = 3$ in., $i = 2$ in., $m = 15$, $P = 30{,}000$.

The neutral axis depth, $n = 11{\cdot}5$ in., is calculated by Formula (111).

$$P = (\tfrac{1}{2} \times 482 \times 10 \times 11{\cdot}5) + (5344 \times 1{\cdot}2) - (4086 \times 1) = 30{,}000 \text{ (lb.)} \qquad (107)$$

$$B = [\tfrac{1}{2} \times 482 \times 10 \times 11{\cdot}5(10 - 3{\cdot}8)] + [5344 \times 1{\cdot}2 \times 7] + [4086 \times 1 \times 8] = 250{,}000 \text{ (in.-lb.)} \qquad (108)$$

$$rs' = \frac{(7 - 10 + 11{\cdot}5)}{11{\cdot}5} \times 15 \times 482$$

$$= \frac{(11{\cdot}5 - 3)}{11{\cdot}5} \times 15 \times 482 = 5344 \text{ (lb./in.}^2\text{)} \qquad (109)$$

$$rs = \frac{(8 + 10 - 11{\cdot}5)}{11{\cdot}5} \times 15 \times 482$$

$$= \frac{20 - (2 + 11{\cdot}5)}{11{\cdot}5} \times 15 \times 482 = 4086 \text{ (lb./in.}^2\text{)} \qquad (110)$$

$$rc' = \frac{30{,}000 \times 11{\cdot}5}{[\tfrac{1}{2}\times 10\times 11{\cdot}5^2]+[15\times 1{\cdot}2(11{\cdot}5-3)]-[15\times 1\times 6{\cdot}5]}$$

$$= 482 \text{ (lb./in.}^2\text{)} \qquad (112)$$

$$rc = \frac{(20 - 11{\cdot}5)}{11{\cdot}5} \times 482 = 356 \text{ (lb./in.}^2\text{)} \qquad (113)$$

SUMMARY OF FORMULÆ FOR MEMBERS UNDER COMBINED STRESSES

MEMBERS WHERE THE RESULTANT STRESSES ARE COMPRESSIVE ON ONE SIDE OF THE AXIS AND TENSILE ON THE OTHER SIDE OF THE AXIS

Formulæ Based upon the Theory of Simple Flexure.
(Simplified Forms for Symmetrical Sections.)

$$P = \tfrac{1}{2}rc'\cdot b\cdot n + (rs' - rs)As \qquad (107a)$$

$$B = \tfrac{1}{2}rc'\cdot b\cdot n(\tfrac{1}{2}D - \tfrac{1}{3}n) + (rs' + rs)As\cdot as \qquad (108a)$$

$$rs' = \frac{as - \tfrac{1}{2}D + n}{n}\cdot m\cdot rc'$$

$$= \frac{(n - i)}{n}\cdot m\cdot rc' \qquad (109a)$$

$$rs = \frac{as + \tfrac{1}{2}D - n}{n}\cdot m\cdot rc'$$

$$= \frac{D - (i + n)}{n}\cdot m\cdot rc' \qquad (110a)$$

$$0 = n^3\cdot\tfrac{1}{6}P - n^2(\tfrac{1}{4}P\cdot D - \tfrac{1}{2}B)$$

$$+ 2n\cdot B\cdot m\cdot As/b - m\cdot As/b(B\cdot D + 2P\cdot as^2) \qquad (111a)$$

$$rc' = \frac{P}{\tfrac{1}{2}b\cdot n + m\cdot As\left(\dfrac{2n - D}{n}\right)} \qquad (112a)$$

$$rc = \frac{D - n}{n}\cdot rc' \qquad (113a)$$

NUMERICAL EXAMPLES

MEMBERS WHERE THE RESULTANT STRESSES ARE COMPRESSIVE ON ONE SIDE OF THE AXIS AND TENSILE ON THE OTHER SIDE OF THE AXIS

(*Symmetrical Section.*)

DATA.—$As = 1$ sq. in. ($As' = As = 1$ sq. in. above the central axis, not to be used in this series of equations), $as = 8$ in. (above or below the central axis), $b = 10$ in., $D = 20$ in., $i = 2$ in. (below the top and above the bottom of the members), $m = 15$, $n = 11\cdot65$ in. (as calculated by (111*a*) on page 235). $P = 30{,}000$ lb.

$$P = (\tfrac{1}{2}\times 480\times 10\times 11\cdot65)+(5964-3924)\times 1 = 30{,}000 \text{ (lb.)} \qquad (107a)$$

$$B = [\tfrac{1}{2}\times 480\times 10\times 11\cdot65(10-\tfrac{1}{3}\times 11\cdot65)] + [(5964+3924)\times 1\times 8] = 250{,}000 \text{ (in.-lb.)} \qquad (108a)$$

$$rs' = \frac{8-10+11\cdot65}{11\cdot65}\times 15\times 480$$

$$= \frac{11\cdot65-2}{11\cdot65}\times 15\times 480 = 5964 \text{ (lb./in.}^2) \qquad (109a)$$

$$rs = \frac{8+10-11\cdot65}{11\cdot65}\times 15\times 480$$

$$= \frac{20-(2+11\cdot65)}{11\cdot65}\times 15\times 480 = 3924 \text{ (lb./in.}^2) \qquad (110a)$$

$$rc' = \frac{30{,}000}{\tfrac{1}{2}\times 10\times 11\cdot65+15\times 1\left(\dfrac{2\times 11\cdot65-20}{11\cdot65}\right)}$$

$$= 480 \text{(lb./in.}^2) \qquad (112a)$$

$$rc = \frac{20-11\cdot65}{11\cdot65}\times 480 = 344 \text{ (lb./in.}^2) \qquad (113a)$$

SUMMARY OF FORMULÆ FOR MEMBERS UNDER COMBINED STRESSES

MEMBERS WHERE THE RESULTANT STRESSES ARE COMPRESSIVE ON ONE SIDE OF THE AXIS AND TENSILE ON THE OTHER SIDE OF THE AXIS

Formulæ Based upon the Theory of Simple Flexure.

(Further Simplified Forms for Symmetrical Sections.)

$$P = [\tfrac{1}{2}rc'\cdot n_1 + r(rs' - rs)]b\cdot D \quad . \quad . \quad . \quad . \quad (107b)$$

$$B = [\tfrac{1}{2}rc'\cdot n_1(\tfrac{1}{2} - \tfrac{1}{3}n_1) + r(rs' + rs)as_1]b\cdot D^2 \quad . \quad . \quad (108b)$$

$$rs' = \frac{(n_1 - i_1)}{n_1}\cdot m\cdot rc' \quad . \quad . \quad . \quad . \quad . \quad . \quad (109b)$$

$$rs = 1\,\frac{(i_1 + n_1)}{n_1}\cdot m\cdot rc' \quad . \quad . \quad . \quad . \quad . \quad (110b)$$

$$rc' = \frac{P}{\left[\tfrac{1}{2}n_1 + m\cdot r\left(\dfrac{2n_1 - 1}{n_1}\right)\right]b\cdot D} \quad . \quad . \quad . \quad . \quad (112b)$$

$$rc = \frac{(1 - n_1)}{n_1}\cdot rc' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113b)$$

(Equations for Diagrams giving Values of n_1.)

$$\frac{B}{P} = e = \frac{-n^3 + 3n^2\cdot\tfrac{1}{2}D + 12m\cdot As\cdot as^2/b}{3n^2 + 12n\cdot m\cdot As/b - 6D\cdot m\cdot As/b} \quad . \quad . \quad (114)$$

$$\frac{B}{P} = e = \left[\frac{-n_1^3 + 1\tfrac{1}{2}n_1^2 + 12m\cdot r\cdot as_1^2}{3n_1^2 + 12n_1\cdot m\cdot r - 6m\cdot r}\right]D \quad . \quad . \quad . \quad (114a)$$

$$\frac{B}{P\cdot D} = e_1 = \frac{-n_1^3 + 1\tfrac{1}{2}n_1^2 + 28{\cdot}8r}{3n_1^2 + 180n_1\cdot r - 90r} \quad . \quad . \quad . \quad . \quad (115)$$

NUMERICAL EXAMPLES

MEMBERS WHERE THE RESULTANT STRESSES ARE COMPRESSIVE ON ONE SIDE OF THE AXIS AND TENSILE ON THE OTHER SIDE OF THE AXIS

(*Symmetrical Sections.*)

DATA.—$r = As/(b \cdot D) = 0{\cdot}005$ ($r' = r = 0{\cdot}005$ above the central axis, not to be used in these equations), $as_{\prime} = as/D = 0{\cdot}4$, $b = 10$ in., $D = 20$ in. [$D/D = 1$ used in (110b) and (113a)], $i_{\prime} = i/D = 0{\cdot}1$ (below the top and above the bottom of the member), $m = 15$, $n_{\prime} = n/D = 0{\cdot}58$, $P = 30{,}000$ lb.

$$P = [\tfrac{1}{2}(480) \times {\cdot}58 + {\cdot}005(5964 - 3924)]b \cdot D$$
$$= [139{\cdot}8 + 10{\cdot}2]b \cdot D$$
$$= 150 b \cdot D \qquad (107b)$$

$$B = \left[139{\cdot}8\left(\tfrac{1}{2} - \frac{{\cdot}58}{3}\right) + {\cdot}005(5964 + 3924){\cdot}4\right] b \cdot D^2$$
$$= [139{\cdot}8 \times {\cdot}305 + 49{\cdot}44 \times {\cdot}4]b \cdot D^2$$
$$= 62{\cdot}5 b \cdot D^2 \qquad (108b)$$

$$rs' = \frac{{\cdot}58 - {\cdot}1}{{\cdot}58} \times 15 \times 480 = 5964 \text{ (lb./in.}^2\text{)} \qquad (109b)$$

$$rs = \frac{1 - ({\cdot}1 + {\cdot}58)}{{\cdot}58} \times 15 \times 480 = 3924 \text{ (lb./in.}^2\text{)} \qquad (110b)$$

$$rc' = 30{,}000\Big/\left[\tfrac{1}{2}({\cdot}58) + 15({\cdot}005)\left(\frac{2({\cdot}58) - 1}{{\cdot}58}\right) b \cdot D\right]$$
$$= 30{,}000/({\cdot}3125 b \cdot D)$$
$$= 96{,}000/b \cdot D$$
$$= 96{,}000/200 = 480 \text{ (lb./in.}^2\text{)} \qquad (112b)$$

$$rc = \frac{1 - {\cdot}58}{{\cdot}58} \times 480 = 344 \text{ (lb./in.}^2\text{)} \qquad (113b)$$

Numerical examples of equations (114) to (115) are given on the succeeding page.

NUMERICAL EXAMPLES

CALCULATIONS FOR DIAGRAM

GIVING NEUTRAL AXIS DEPTH RATIOS FOR DIFFERENT RATIOS OF ECCENTRICITY AND OF REINFORCEMENT

(Symmetrical Sections.)

Values of n_1 and r are assumed and the corresponding values of e_1 are calculated. The results are plotted in curves from which any required value of n_1 can be obtained.

DATA.—$As = 4$ sq. in. ($As' = As = 4$ sq. in. above the central axis not to be used in the equations), $as = 8$ in., $b = 10$ in., $D = 20$ in., $m = 15$, $n = 12$ in.

$$\frac{B}{P} = e = \frac{12^3 + 3(12)^2 \times \frac{1}{2}(20) + 12 \times 15 \times 4 \times 8^2/10}{3(12^2) + 12 \times 12 \times 15 \times 4/10 - 6 \times 20 \times 15 \times 8/10}$$

$$= \frac{1728 + 4320 + 4608}{432 + 864 - 720}$$

$$= \frac{7200}{576} = 12{\cdot}5 \text{ (inches)} \qquad (114)$$

DATA.—$r = As/(b{\cdot}D) = 0{\cdot}02$ ($r' = r = 0{\cdot}02$ not to be used), $as_1 = as/D = 0{\cdot}4$, $b = 10$ in., $D = 20$ in., $m = 15$, $n_1 = n/D = 0{\cdot}6$.

$$\frac{B}{P} = e = \left[\frac{-{\cdot}6^3 + 1\tfrac{1}{2}({\cdot}6)^2 + 12 \times 15 \times {\cdot}02 \times {\cdot}4^2}{3({\cdot}6)^2 + 12 \times {\cdot}6 \times 15 \times {\cdot}02 - 6 \times 15 \times {\cdot}02}\right] D$$

$$= \left[\frac{-{\cdot}216 + {\cdot}54 + {\cdot}576}{1{\cdot}08 + 2{\cdot}16 - 1{\cdot}80}\right] D$$

$$= 0{\cdot}625\, D = 12{\cdot}5 \text{ (inches)} \qquad (114a)$$

Dividing by D, and inserting $28{\cdot}8 = 12m{\cdot}as_1^2$, $108 = 12n_1{\cdot}m$, and $90 = 6m$, we have

$$\frac{B}{P{\cdot}D} = e_1 = \frac{-{\cdot}6^3 + 1\tfrac{1}{2}({\cdot}6)^2 + 28{\cdot}8 \times {\cdot}02}{3({\cdot}6)^2 + (180 \times {\cdot}6 - 90){\cdot}02}$$

$$= \frac{-{\cdot}216 + {\cdot}54 + 28{\cdot}8 \times {\cdot}02}{1{\cdot}08 + 18 \times {\cdot}02}$$

$$= 0{\cdot}625 \qquad (115)$$

CALCULATION OF CURVES FOR DIAGRAM

VALUES OF e_i FOR DIFFERENT VALUES OF n_i AND r.

Taking $m = 15$, and $as_i = 0{\cdot}4$ as constants, there are only two variables in equation (115), by the aid of which values of $e_i = B/P{\cdot}D$ for different values of n_i and r can readily be calculated.

CALCULATION OF e_i CURVE FOR $n_i = 0{\cdot}6$ and $r = 0{\cdot}001$ to $0{\cdot}005$.

By (115), $e_i = {\cdot}324 + 28{\cdot}8r/(1{\cdot}08 + 18r)$.
Therefore we have

r	Numerator.		Denominator.		e_i
	$28{\cdot}8r$	$324 + 28{\cdot}8r$	$18r$	$1{\cdot}08 + 18r$	
0·001	0·0288	0·3528	0·018	1·098	0·3213
·002	·0576	·3816	·036	1·116	·3419
·003	·0864	·4104	·054	1·134	·3619
·004	·1152	·4391	·072	1·152	·3814
·005	·1440	·4680	·090	1·170	·4000

Taking $m = 15$ and $as_i = 0{\cdot}408$ as constants, the calculation of curves is simplified although $as_i = 0{\cdot}408$ is not so convenient as $e_i = 0{\cdot}4$ for general use.

CALCULATION OF e_i CURVE FOR $n_i = 0{\cdot}6$ and $r = 0{\cdot}001$ to $0{\cdot}005$.

By (115), $e_i = {\cdot}324 + 30r/(1{\cdot}08 + 18r)$.
Therefore we have

r	Numerator.		Denominator.		e_i
	$30r$	$\cdot324 + 30r$	$18r$	$1{\cdot}08 + 18r$	
0·001	0·03	0·354	0·018	1·098	0·3223
·002	·06	·384	·036	1·116	·3442
·003	·09	·414	·054	1·134	·3655
·004	·12	·444	·072	1·152	·3854
·005	·15	·474	·090	1·170	·4052

To provide data for plotting a complete diagram on any predetermined basis, results must be calculated with a sufficient number of values for n_i, each with values of r ranging from 0 upwards.

INDIRECT SOLUTION OF CUBIC EQUATION
FOR NEUTRAL AXIS DEPTH.

(Unsymmetrical Sections.)

DATA.—The data applying to the examples on page 227 are reprinted here for the convenience of the reader as follows: $As = 1$ sq. in., $as = 8$ in., $As' = 1{\cdot}2$ sq. in., $as' = 7$ in., $B = 250{,}000$ in.-lb., $b = 10$ in., $D = 20$ in., $m = 15$, $P = 30{,}000$ lb.

Substituting these values in equation (111), we have—

$$(a)\quad n^3 \times \tfrac{1}{6} \times 30{,}000 \quad = \quad 5{,}000\, n^3$$

$$(b)\quad -\, n^2(\tfrac{1}{4} \times 30{,}000 \times 20 - \tfrac{1}{2} \times 250{,}000) \quad = \quad 25{,}000\, n^2$$

$$(c)\quad +\, n \times 15/10[250{,}000(1{\cdot}2+1) - 30{,}000(1{\cdot}2 \times 7 - 1 \times 8)] \quad = \quad 807{,}000\, n$$

$$(d)\quad -\,250{,}000 \times 15/10[1{\cdot}2 \times 3 + 1 \times 18] - 30{,}000 \times 15/10\,[1 \times 8 \times 18 - 1{\cdot}2 \times 7 \times 3] \quad = \quad 13{,}446{,}000$$

Dividing the results for terms (a) to (d) by 5,000, we obtain

$$n^3 - 5n^2 + 160n - 2690 = 0$$

As a first trial, assume $n = 10$, giving

$$1000 - 500 + 1600 - 2690 = -\ 590$$

A second trial, with $n = 12$ gives

$$1728 - 720 + 1920 - 2690 = +\ 238$$

Therefore, it is evident that the value of n must be between 10 and 12 and nearer to 12 than to 10.

A third trial, with $n = 11$ results in

$$1331 - 605 + 1760 - 2690 = -\ 204$$

A fourth trial, with $n = 11{\cdot}5$ gives the result

$$1520 - 661 + 1840 - 2690 = +\ \ 9$$

This is near enough for all practical purposes, and further trial is unnecessary and the value $n = 11{\cdot}5$ is adopted in the calculations on the preceding page.

As a matter of interest, it may be mentioned that a fifth trial, with $n = 11{\cdot}48$, provides an exact solution.

INDIRECT SOLUTION OF CUBIC EQUATION

FOR NEUTRAL AXIS DEPTH.

(*Symmetrical Sections.*)

DATA.—The following date are also employed for the numerical examples on page 229: $As = 1$ sq. in. (with $As' = As = 1$ sq. in. above the central axis, not to be used in the equation), $as = 8$ in., $B = 250{,}000$ in.-lb., $b = 10$ in., $D = 20$ in., $m = 15$, $P = 30{,}000$.

Substituting these values in equation (111*a*), we have

(*a*) $n^3 \cdot \frac{1}{6} \times 30{,}000 \qquad = 5{,}000\, n^3$

(*b*) $- n^2(\frac{1}{4} \times 30{,}000) \times 20 - \frac{1}{2} \times 250{,}000) \qquad = 25{,}000\, n^2$

(*c*) $+ 2n \times 250{,}000 \times 15 \times 1/10 \qquad 150\, n$

(*d*) $- 15 \times 1/10(250{,}000 \times 20 + 2 \times 30{,}000 \times 8^2) = 2{,}650$

Dividing the results calculated for terms (*a*), (*b*), (*c*), and (*d*) by 5,000, we obtain—

$$n^3 - 5n^2 + 150n - 2650 = 0$$

Successive trials give results as follows

$n = 10 \qquad 1000 - 500 + 1500 - 2650 = - 650$

$n = 12 \qquad 1728 - 720 + 1800 - 2650 = + 158$

$n = 11{\cdot}5 \qquad 1520 - 661 + 1725 - 2650 = - 66$

$n = 11{\cdot}6 \qquad 1560 - 672 + 1740 - 2650 = - 22$

$n = 11{\cdot}65 \qquad 1581 - 679 + 1747 - 2650 = - 1$

In cases where an exact value for n is required the results of a few trials can be plotted so as to provide a curve showing the value which, inserted in (111*a*), will give 0 as the result.

For ordinary calculations, values of n can be obtained readily from the diagram reproduced as Fig. 96.

MEMORANDA

CHAPTER XI

EQUIVALENT AREA; INERTIA AND BENDING MOMENTS; WORKING STRESSES

EQUIVALENT AREA

Definition.—The *equivalent area*, or, as it is sometimes termed, the *transformed area*, of a reinforced concrete member is an imaginary or theoretical area equivalent to the sectional area of the concrete plus m times the sectional area of the steel.

Let Ae = equivalent area, A = total area of the member, ΣAs = total area of the steel, and m = modular ratio Es/Ec.

Then

$$\begin{aligned} Ae &= A - \Sigma As + m \cdot \Sigma As \quad . \quad . \quad . \\ &= A + m \cdot \Sigma As - \Sigma As \\ &= A + (m-1)\, \Sigma As \quad . \quad . \quad . \quad (116) \end{aligned}$$

Fig. 98 represents the section of a member 10 in. square, where the total area of the steel is made up of four bars each of 1 sq. in. area.

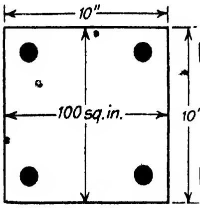


FIG. 98

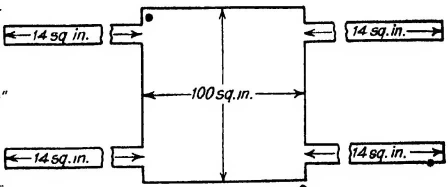


FIG. 99

By the first form of (116) $Ae = 100 - 4 + 15 \times 4 = 96 + 60 = 156$ (sq. in.). Fig. 99 represents the equivalent area Ae, calculated by the final form of (116), taking $m = 15$, as follows—

$$Ae = 100 + (15 - 1) \times 4 = 156 \text{ (square inches)}$$

In this example, the total area of (100 sq. in.) is supplemented by the equivalent area of steel (4×14) $= 56$ (sq. in.). Each bar of steel has an area equal to $\frac{1}{4}\Sigma As$, and each of the projecting wings in Fig. 99 represents an 'area' equal to 14 sq. in. of concrete.

Formula (116) is a general equation in which the symbol ΣAs may be replaced by some analagous symbol in the case of any class of member where the total area of steel is not usually denoted by the symbol ΣAs.

Thus, in formulæ for beams, either A or $(A + Ac)$ may denote the total area of steel; in vertical compression members, Av = total area of vertical steel (Fig. 100); in

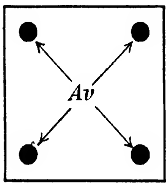


FIG. 100

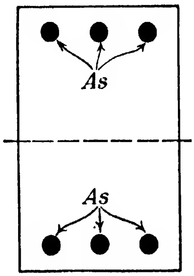


FIG. 101

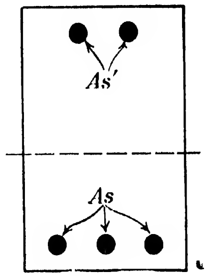


FIG. 102

members subject to combined stresses, $(As + As)$ = total area of steel if, as in Fig. 101, the reinforcement is arranged symmetrically above and below the centroidal axis; and in members of the same class $(As + As')$ = total area of steel if, as in Fig. 102, the reinforcement is unsymmetrical above and below the centroidal axis.

From the practical standpoint it does not matter whether we use As, Av, $(As + As)$, $As + As'$), or any other symbols in (116) so long as the total area of steel is represented and taken into account.

By the employment of the symbol Σr to denote the total ratio of steel, we can write (116) in the simplified form

$$Ae = [1 + (m - 1)\Sigma r]A \quad . \quad . \quad . \quad (116a)$$

Here Σr is the ratio of the total area of steel to the total area (or, in the case of a beam, to the effective area) of the member under consideration.

The ratios of steel for different classes of members are defined in Table VIII, where also will be found several variations of the general equation (116) for the equivalent area of beam, compression member and combined stress member sections. This table, which includes corresponding variations of the general equation for inertia moments, is given with the objects of affording guidance in the use of the symbols denoting ratios and areas of reinforcement, and of collecting the formulæ together so that they may be readily compared.

INERTIA MOMENT AND DERIVED RATIOS

Inertia Moment.—As the inertia moment, I, of a body is the sum of the products obtained by multiplying each element of mass by the square of its distance from the centroidal axis of the body, the inertia moment, Ie, of a reinforced concrete section in terms of the equivalent area, Ae, is necessarily more complex than the inertia moment of a homogeneous section; and includes the inertia moment, Ic, of the concrete, and the inertia moment, Is, of the steel.

The following equations apply to forms of section in frequent use, the reinforcing bars being assumed to be arranged as in Figs. 103 to 105.

Rectangular Section with Symmetrical Reinforcement.—In this case (see Fig. 103) we have, by the familiar rule,

$$Ic = \tfrac{1}{12}b \cdot D^3$$

which, as $ac = \frac{1}{2}D$, may be written

$$Ic = \tfrac{1}{3}b \cdot D \cdot ac^2$$
$$= \tfrac{1}{3}A \cdot ac^2$$

For the inertia moment of the total area of steel, we have

$$Is = (m-1)\Sigma As \cdot as^2$$

TABLE VIII

Comparison of Ratios of Steel, Equivalent Areas, and Inertia Moments for Different Classes of Members (Rectangular Section).

Classes of of Members.	Ratio of Steel	Equivalent Area.	Inertia Moment.
All Classes	$r = \Sigma As/A$	$Ae = A - (m-1)\Sigma As$	$Ie = Ic + (m-1)\Sigma As \cdot as^2$ *Symmetrical* $Ie = Ic + (m-1)[As \cdot as^2 + As' \cdot as'^2]$ *Unsymmetrical*
Beams*	$r = A/Ab$	$Ae = Ab + (m-1)A$	$Ie = \frac{1}{3}b[n^3 + (d-n)^3] + (m-1)A(d-n)^2$
	$rc = Ac/Ab$	$Ae = Ab - (m-1)Ac$	$Ie = \frac{1}{3}b[n^3 - (d-n)^3] + (m-1)Ac(n-i)^2$
	$(r + rc) = (A + Ac)/Ab$	$Ae = Ab - (m-1)(A + Ac)$	$Ie = \frac{1}{3}b[n^3 - (d-n)^3] - (m-1)[A(d-n^2) + Ac(n-i)^2]$
Compression Members	$r = Av/A$	$Ae = A - (m-1)Ar$	$Ie = \frac{1}{3}A \cdot ae^2 + (m-1)Ar \cdot as^2$
Combined Stress Members	$r = As/A$	$Ae = A + (m-1)As$	$Ie = \frac{1}{3}b(ae^3 + ae'^3) + (m-1)As.as^2$
	$r' = As'/A$	$Ae = A - (m-1)As'$	$Ie = \frac{1}{3}b(ae^3 - ae'^3) - (m-1)As' \cdot as'^2$
	$(r + r') = (As + As')/A$	$Ae = A + (m-1)(As - As')$	$Ie = \frac{1}{3}b(ae^3 + ae'^3) - (m-1)[As \cdot as^2 + As' \cdot as'^2]$
	$(r + r) = (As + As)/A$	$Ae = A - (m-1)(As + As)$	$Ie = \frac{1}{3}A \cdot ae^2 + (m-1)(As + As)as^2$

* In the case of beams, Ab = effective area of beam, and A = area of tension reinforcement.

and for the inertia moment of the reinforced section

$$Ie = Ic + Is$$

$$= [\tfrac{1}{8}A \cdot ae^2] + [(m-1)\Sigma As \cdot as^2] \quad . \quad (117)$$

Rectangular Section with Unsymmetrical Reinforcement.—Where the reinforcement is unsymmetrical, as in Fig. 104, we have

$$Ic = \tfrac{1}{3}b(ae^3 + ae'^3)$$

and

$$Is = (m-1)\ (As \cdot as^2 + As' \cdot as'^2)$$

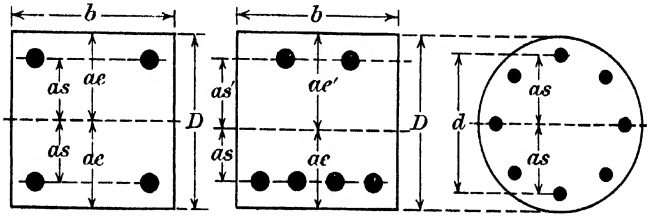


FIG. 103 FIG. 104 FIG. 105

Then, for the inertia moment of the reinforced section we have

$$Ie = [\tfrac{1}{3}b(ae^3 + ae'^3)] + (m-1)[(As \cdot as^2 + As' \cdot as'^2)]$$

(118)

Circular Section with Symmetrical Reinforcement.—In this case (see Fig. 105) we have

$$Ic = \tfrac{1}{16}A \cdot D^2$$

and, denoting $(as + as)$ by the symbol d,

$$Is = \tfrac{1}{8}(m-1)\ \Sigma As \cdot d^2$$

Therefore for the inertia moment of the reinforced section, we obtain

$$Ie = [\tfrac{1}{16}A \cdot D^2] + [\tfrac{1}{8}(m-1)\Sigma As \cdot d^2] \qquad (119)$$

Formulæ (117) to (119) are general equations applicable to all classes of reinforced concrete members of rectangular

and circular cross sections for which it is necessary to calculate inertia moments, and as explained in connection with Formula (116) the symbols may be altered as desired in order to comply with any special system of notation adopted for compression members or members subject to combined stresses. Variations of the formulæ embodying the notation employed for beams, compression members, and combined stress members are given in Table VIII.

Modulus of Section.—The modulus of a section is a measure of the strength of the section which is more generally employed in ordinary structural design than in reinforced concrete practice.

In the familiar equation for the resistance moment, $R = fI/h$, the quantity I/h is the modulus of the section, and may be described as the inertia moment expressed in terms of h. Denoting this ratio by the symbol M, we have the general expression

$$M = I/h$$

Where h = *height* of the extreme fibres above the neutral axis of a homogeneous member, this dimension being equal to the distance between the neutral axis and the extreme fibres below the same axis.

The equation applies to all sections which are symmetrical about the neutral or centroidal axis, and can be applied to the section of a reinforced concrete beam or other member with the substitution of the symbols Ie for I and either n or ac for h.

Then, for the section of beam, we have

$$M = Ie/n$$

and for the section of a compression member, or a member subject to combined stresses, where $ae = \frac{1}{2}D$ as in Fig. 106, we have

$$M = Ie/ae$$

In the case of reinforced concrete beams and members subject to combined stresses, ordinary methods of computing and applying the modulus of section do not satisfy requirements.

As we have shown when dealing with beam and other formulæ, the equations suitable for homogeneous materials require considerable amplification so as to fit them for taking into account the various factors occurring in reinforced concrete practice. Similarly, to render the modulus of section suitable for general employment in reinforced concrete formulæ, qualifications of the ratio expressed by the symbol M are necessary, particularly in equations for sections unsymmetrical about the neutral or the centroidal axis.

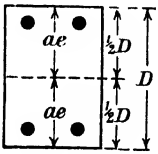


FIG. 106

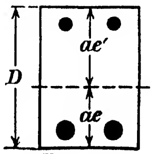


FIG. 107

Leaving beams out of account, let us consider a section such as that represented in Fig. 107, which may be taken as the cross section of any class of member subject to combined stresses.

In order to provide qualified forms of the general symbol M, suitable for use in combined stress formulæ, the modulus of section must be suitably derived from the inertia moment stated in one or other of the forms given in Table VIII.

For instance, the modulus $M = Ie/ae$ for a symmetrical section must be replaced by $M' = Ie/ae'$, and $M = Ie/ae$, where ae' and ae are the arms of the extreme fibres above and below the centroidal axis, respectively.

Again, the arms of the steel used as reinforcement, above and below the centroidal axis, demand the additional symbols $Ms' = Ie/as'$, and $Ms = Ie/as$.

In the ordinary beam equation $f = B/M$, we have only the flexural stress f to determine, but in reinforced concrete formulæ for beams and members subject to combined direct and flexural stresses, we have to determine the values of tensile and compressive stresses due to flexure on either side of the neutral or the centroidal axis.

The following flexural stresses have to be considered in a combined stress member—

fc' = flexural stress in concrete extreme fibres above centroidal axis.
fc = " " " extreme fibres below centroidal axis.
fs' = " " steel above centroidal axis.
fs = " " " below centroidal axis.

Therefore, to provide for determination of these stresses, the simple equation $f = B/M$ must be replaced by

$$fc' = B/M' = B/(Ic/ac')$$
$$fc = B/M = B/(Ic/ac)$$
$$fs' = B/Ms' = B/(Ic/as')$$
$$fs = B/Ms = B/(Ic/as)$$

On reference to Chapter X, it will be seen that these expressions, in their second form, are included in equations (102) to (105), and as explained in the same chapter the first form can be substituted if thought desirable. From the foregoing equations we can derive

$$M = Ic/ac \quad \ldots \quad (120)$$
$$M' = Ic/ac' \quad \ldots \quad (121)$$
$$Ms = Ic/as \quad \ldots \quad (122)$$
$$Ms' = Ic/as' \quad \ldots \quad (123)$$

Gyration Radius.—This is another derivative from the inertia moment of a section, and it represents the distance of the *centre of gyration* from the axis about which the inertia moment has been taken. The centre of gyration is a point such that if the area of the section could be concentrated there the moment of inertia would be the same as that of the actual area.

The relations between the inertia moment, I, and the gyration radius, g, can be simply explained as follows—

By formula (117) the inertia moment for a plain concrete is—

$$I = \tfrac{1}{3}A \cdot ae^2$$
$$= A(\tfrac{1}{3}ae^2)$$

where A = area of the section and $(\tfrac{1}{3}ae^2) = g^2$.

Consequently for a homogeneous section we have

$$I = A \cdot g^2$$
$$g^2 = I/A$$
$$g = \sqrt{(I/A)}$$

For a reinforced concrete section, the inertia moment of the equivalent area must be taken and equation (130) becomes

$$g = \sqrt{(Ie/Ae)} \qquad . \quad . \quad . \quad (124)$$

BENDING MOMENTS

Rules for the computation of bending moments for beams under all conditions of loading and support are to be found in standard text-books on applied mechanics and the strength of materials. Attention is therefore confined to the statement of a few rules which will be found generally useful in reinforced concrete practice.

Independent Beams.—The rules given below, taken from the London County Council Regulations, are applicable to ordinary reinforced concrete construction in cases where precise and laborious calculations are unnecessary.

TABLE IX.
BENDING MOMENTS FOR BEAMS.
CL = Concentrated Load. UL = Uniformly-distributed Load.

Load Conditions.	*Mode of Supporting or Fixing Ends.*	*Maximum Bending Moment.*
CL (at free end) *UL*	One fixed, one free " "	$B = W \cdot l$ (at fixed end) $B = W \cdot l/2$ " "
CL (at centre) *UL*	Both freely supported " " "	$B = W \cdot l/4$ (at centre) $B = W \cdot l/8$ " "
CL (at centre) *UL*	Both fixed " "	$B = W \cdot l/8$ (at centre and ends) $B = W \cdot l/12$ " "
UL	One fixed, one freely supported	$B = W \cdot l/8$ (at fixed end) $B = W \cdot l/14$ (at $\tfrac{3}{8}$ of span from freely supported end)

SUMMARY OF MISCELLANEOUS FORMULÆ

EQUIVALENT AREA

General Equation.

$$Ae = A + (m-1)\Sigma As \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

$$= [1 + (m-1)\Sigma r]A \quad . \quad . \quad . \quad . \quad . \quad (116a)$$

Variations of the general equation are given in Table VIII for Beams, Compression Members, and Members under Combined Stresses.

INERTIA MOMENTS

General Equations.

Rectangular Section Symmetrical about Axis.

$$Ie = [\tfrac{1}{3}A \cdot ae^2] + [(m-1)\Sigma As \cdot as^2] \quad . \quad . \quad . \quad . \quad (117)$$

Rectangular Section Unsymmetrical about Axis.

$$Ie = [\tfrac{1}{3}b(ae^3 + ae'^3)] + [(m-1)(As \cdot as^2 + As' \cdot as'^2)] \quad . \quad (118)$$

Variations of the general equations are given in Table VIII for Beams, Compression Members, and Members under Combined Stresses.

Circular Section Symmetrical about Axis.

$$Ie = [\tfrac{1}{16}A \cdot D^2] + [\tfrac{1}{8}(m-1)\Sigma As \cdot d^2] \quad . \quad . \quad . \quad . \quad (119)$$

MODULUS OF SECTION

$$M = Ie/ae \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

$$M' = Ie/ae' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$Ms = Ie/as \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

$$Ms' = Ie/as' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

GYRATION RADIUS

$$g = \sqrt{Ie/ae} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

NUMERICAL EXAMPLES

EQUIVALENT AREA

Rectangular Section.

DATA.—$b = 10$ in., $D = 20$ in., $A = b \cdot D = 200$ sq. in., total area of steel $= \Sigma As = 4$ sq. in., $\Sigma r = 4/200 = 0 \cdot 02$, $m = 15$.

$$Ae = 200 + 14 \times 4 = 256 \text{ (sq. in.)} \quad . \quad . \quad . \quad (116)$$

$$Ae = [1 + 14 \times \cdot 02]200 = 256 \text{ (sq. in.)} \quad . \quad . \quad (116a)$$

INERTIA MOMENTS

Rectangular Section Symmetrical about Centroidal Axis.

DATA.—$b = 10$ in., $D = 20$ in., $ac = 10$ in., $as = 8$ in., total area of steel $= \Sigma As = 4$ sq. in., $m = 15$.

$$Ie = \tfrac{1}{3} \times 200 \times 10^2 + 14 \times 4 \times 8^2$$
$$= 6666 + 3584 = 10{,}250 \text{ (inch units)} \quad . \quad . \quad (117)$$

Rectangular Section Unsymmetrical about Centroidal Axis.

DATA.—$b = 10$ in., $D = 20$ in., $ac = 10 \cdot 5$ in., $ac' = 9 \cdot 5$ in., $as = 9 \cdot 5$ in., $as' = 7 \cdot 5$ in., $As = 1$ sq. in., $As' = 2$ sq. in., $m = 15$.

$$Ie = \tfrac{1}{3} \times 10\,(10 \cdot 5^3 + 9 \cdot 5^3) + 14(1 \times 9 \cdot 5^2 + 1 \times 7 \cdot 5^2)$$
$$= 6716 + 2838 = 9554 \text{ (inch units)} \quad . \quad . \quad . \quad (118)$$

Circular Section Symmetrical about Central Axis.

DATA.—$A = 201$ sq. in., $\Sigma As = 9$ sq. in., $d = (as + as) = 13$ in., D (overall diameter) $= 16$ in., $m = 15$.

$$Ie = \tfrac{1}{16} \times 201 \times 16^2 + \tfrac{1}{8} \times 14 \times 9 \times 13^2$$
$$= 3216 + 2661 = 5877 \text{ (inch units)} \quad . \quad (119)$$

It should be noted that in the case of beams with fixed ends, the condition of fixity is rarely attained in structural work to such extent as to justify the use of theoretical rules without qualification.

The French Commission du Ciment Armé and some other authorities recommend that if the fixity of a beam under uniformly distributed loading be partial, the centre bending moment shall be determined by the equation $B = W \cdot l/10$. But as this involves a theoretical value of $B = W \cdot l/40$ at the ends, many authorities consider it prudent to take a uniform value such as that given by $B = W \cdot l/10$, or $B = W \cdot l/12$, as applicable to the ends and the centre. It should be recognized, however, that this value cannot obtain simultaneously at the two points in a beam.

Continuous Beams.—In reinforced concrete construction of ordinary character it is rarely necessary to enter into minute calculation of the bending moments in beams continuous over two or more spans.

An approximate rule in very general employment for the design of beams continuous over three or more spans is to take $B = +W \cdot l/12$ at the centre of interior spans and $B = -W \cdot l/12$ at intermediate supports.

The London County Council Regulations give the following approximate rules for maximum bending moments due to distributed loading over the approximately equal spans of continuous beams-

Near middle of end span . .	$B = +W \cdot l/10$
At support next end support . .	$B = -W \cdot l/10$
At middle of interior spans . .	$B = +W \cdot l/12$
At other interior supports . .	$B = -W \cdot l/10$

Rectangular Slabs Supported along Four Edges.—Theories have been propounded by Bach, Grashof, Rankine, and others for the bending moments in rectangular slabs supported or fixed along all four edges, but no absolutely satisfactory theory has yet been evolved.

At present, the Grashof-Rankine theory appears to be

the best for adoption, and in the form recommended by the French Commission it provides for reductions of bending moments in accordance with the coefficient—

$$\frac{1}{1 + 2\left(\frac{b}{l}\right)^4}$$

where b = breadth of the slab and l = length of the slab.

Hence the proportion of the total load W assumed to be carried in the direction of the breadth of the slab is

$$\frac{1}{1 + 2\left(\frac{b}{l}\right)^4} \cdot W \quad . \quad . \quad . \quad . \quad (a)$$

and the proportion of the load assumed to be carried in the direction of the length of the slab is

$$\frac{1}{1 + 2\left(\frac{l}{b}\right)^4} \cdot W \quad . \quad . \quad . \quad . \quad (b)$$

In applying the Grashof-Rankine theory, the total load on a slab is multiplied by the coefficients as in equations (a) and (b), and the bending moments are calculated in the usual manner. The reinforcement, which must be disposed in two directions at right angles to each other and parallel to the sides of the slab, is then proportioned to provide the necessary resistance.

The designer should bear in mind the possibility of reverse flexure in some spans of continuous slabs as a result of unequal loading, and make such provision as may be required for resistance to tension by the use of reinforcement near the compression surface of the slab.

Slabs Forming Beam Flanges.—In structures where slabs are in monolithic connection with beams spaced at short distances apart, they may be regarded as constituting compression flanges which convert rectangular beams into beams of T-section.

Some designers take the flange width for such beams as equal to the spacing between the centres of the ribs of the

beams. This practice, however, is not universally approved without qualification.

The R.I.B.A. Report says: "The whole of the slab cannot in general be considered to form part of the upper flange of the T-beams. The width b of the upper flange may be assumed to be not greater than one-third the span of the beams, or more than three-fourths of the distance from centre to centre of the reinforcing ribs or more than fifteen times the thickness of the slab. The width br of the rib should not be less than one-sixth of the width b of the flange."

The American Joint Committee recommend the following rules, which have been adopted, with a slight alteration, in the London County Council Regulations.

Under these rules, the effective breadth of slab must not exceed

(*a*) One fourth of the effective span of the beam.

(*b*) The distance between the centres of the beams.

(*c*) Twelve times the thickness of the slab plus the breadth of the rib of the beam.

Of these stipulations, (*a*) and (*c*) are unnecessarily conservative and might safely be altered to agree with the corresponding recommendations of the R.I.B.A. Committee.

In the case of independent T-beams, where the edges of the compression flange are free, the width of the flange should not be more than four or five times the breadth of the web.

A more conservative rule is to make the flange breadth not more than three times the rib breadth and the flange thickness not less than one-third the depth of the beam.

WORKING STRESSES FOR CONCRETE

Considerable divergence of opinion exists with regard to the working stresses that should be adopted in reinforced concrete design.

The point will be realized fully if the reader will compare the permissible working stresses recommended by Government and municipal authorities and technical committees in this and other countries.

No useful purpose would be served by giving an elaborate series of tables embodying the working stresses laid down by such authorities. Some of them are based upon experimental results relating to concrete of inferior quality as judged by modern standards; others are representative of average values; and others again are of more or less arbitrary character, being intended primarily for the purpose of establishing safeguards to protect the public against the effects of indifferent work by inexperienced designers and constructors.

If working stresses were universally fixed at minimum values with the object of giving reasonable assurance of safety even in the case of inferior work, a very undesirable restriction would be placed upon competent designers and constructors.

The standardization of working stresses on an equitable basis would be a simple matter if concrete were a material whose physical properties could be computed with absolute certainty from the proportionate amounts of its constituents.

Various methods have been put forward for calculating or estimating the ultimate strength of concrete, but as there is no guarantee that calculations or estimates will always be justified by practical results, it may be said in a general way that working stresses are obtained by applying a safety factor to an unknown quantity, or at any rate to one of somewhat uncertain value.

Concrete in Tension.—No allowance should be made for the tensile resistance of concrete, except in special cases where the maximum tensile stress is within safe limits of working stress, which usually range from 25 lb. to 35 lb. per square inch.

Concrete in Compression.—It appears to the author that the most satisfactory method of arriving at working stresses for concrete in compression is that embodied in the recommendations of the R.I.B.A. Committee.

Briefly stated, this method consists in making tests of the concrete to be used for any important work, and in taking the working stress at one-third the ultimate

compressive strength of the test specimens at an age of 28 days.

The general adoption of such a method would at once put a premium on the production of the best qualities of concrete, and give to those capable of obtaining good results, an advantage which they do not enjoy if compelled to work under rules devised with special regard to the inexperienced or unreliable contractor.

Tables X and XI give the working stresses recommended by the R.I.B.A. Committee, the London County Council, the American Joint Committee, and the French Commission. These values are based upon the ultimate strength of the various mixtures at the age of 28 days, as estimated by the authorities named, the safety factors being: R.I.B.A. Committee, 3; American Joint Committee, 3·07; French Commission, 2·38. The L.C.C. stresses have been taken as stated in the Regulations, or calculated by a formula there given.

TABLE X

WORKING STRESSES FOR CONCRETE IN COMPRESSION.

PROPORTIONS OF CONCRETE	1 : 2 : 4	1 : 1·5 : 3	1 : 1 : 2
	lb. per sq. in.	*lb. per sq. in.*	*lb. per sq. in.*
R.I.B.A. Committee :			
Gravel or Hard Stone .	600	—	—
L.C.C. Regulations : . .			
Gravel or Hard Stone .	600	675	750
American Committee :			
Gravel	650	814	975
Hard Stone . . .	715	910	1,070

TABLE XI.

WORKING STRESSES FOR CONCRETE IN COMPRESSION.

PROPORTIONS OF CONCRETE.	1 : 1·75 : 3·5	1 : 1·56 : 3·12	1 : 1·37 : 2·75
	lb. per sq. in.	*lb. per sq. in.*	*lb. per sq. in.*
French Commission :			
Gravel	638	717	798
L.C.C. REGULATIONS :			
Gravel	638	666	694
American Committee :			
Gravel	731	793	854

In connection with safety factors, it should be remembered that as concrete increases in strength by nearly 50 per cent. between the ages of one and four months, and thereafter at a gradually decreasing rate, the values of the factors at an age of four months are considerably greater than those stated above.

Flexural and Direct Compression.—With the exception of the American Joint Committee, the authorities mentioned above recommend the same compressive working stresses for members under flexure as for members under direct compression. The reduced stress proposed by the American Committee refers only to columns without transverse reinforcement, and it may be remarked that the R.I.B.A. Committee, who originally recommended a similarly reduced stress, now adopt equal working stresses for beams and columns.

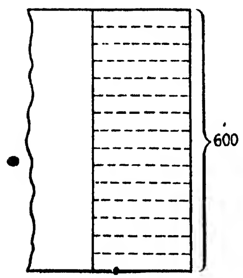


FIG. 108

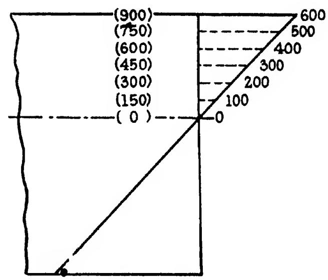


FIG. 109

The question of adopting different working stresses for members subject to flexural and direct compression, respectively, may be considered from two opposite standpoints.

If the extreme fibre stress due to flexural compression, say 600 lb. per square inch, as in Fig. 109, is taken as the basis a reduced working stress would seem to be suggested for members under direct compression where, as represented in Fig. 108, the stress intensity of 600 lb. per square inch is uniform throughout.

On the other hand with direct compressive stress intensity

as a basis, an increased working stress might perhaps be permitted in the case of members subject to flexure as suggested in Fig. 109, for the reason that the maximum stress intensity is attained only at the extreme fibres in the compressed edge, below which the stress intensity rapidily diminishes to zero at the neutral axis.

Concrete in Shear.—The most recent investigations show that the shearing strength of concrete is generally from 50 to 70 per cent. of the compressive strength. Consequently, if resistance to simple shear alone had to be considered, the working stress might reasonably be put at about one-sixth the compressive strength of concrete at 28 days.

As a matter of fact, however, the most important point connected with shearing stress in reinforced concrete beams is that this stress constitutes the only practicable measure of diagonal tension, to which are due most of the failures formerly attributed to shear.

From data available it appears that for concrete of the qualities ordinarily used, a shearing stress of from 100 to 140 lb. per square inch indicates the probable failure by diagonal tension of a beam having no web reinforcement.

Consequently, it is scarcely judicious to recommend working stresses ranging from 50 to 80 lb. per square inch for the shearing stress in cases where this stress is taken as the measure of diagonal tension in reinforced concrete beams.

In the opinion of the author, the working stress should not be put at more than from 25 to 35 lb. per square inch, and it is preferable that the web reinforcement should be designed without taking the shearing resistance of the concrete into account.

Grip or Adhesion between Concrete and Steel.—The recommendation of constant values for grip with concrete of varying proportions is a defect in the R.I.B.A. Report and the L.C.C. Regulations from which the proposals of the French Commission and the American Committee are free. Of the two latter, the American recommendations are preferable as being more closely in accordance than those of the

French Commission with the working stresses justified by experimental results.

Deformed bars giving a mechanical bond may often be used with advantage, particularly in members where insufficient space is afforded for the hooked ends of ordinary plain bars.

TABLE XII.

WORKING STRESSES FOR CONCRETE IN SHEAR AND FOR GRIP OR ADHESION BETWEEN CONCRETE AND STEEL.

Authority.	*Proportions of Concrete.*	*Shear.*	*Grip.*
		lb. per sq. in.	*lb. per sq. in.*
R.I.B.A. Committee .	—	60	100
L.C.C. Regulations .	1 : 2 : 4	60	100 (*bars hooked at both ends*)
	1·2 : 2 : 4	65	
	1·5 : 2 : 4	70	
	2 : 2 : 4	75	60 (*bars otherwise anchored*)
American Committee	1 : 2 : 4	40	80
	1 : 1·5 : 3	50	100
	1 : 1 : 2	60	120
French Commission .	1 : 1·75 : 3·5	56	56
	1 : 1·56 : 3·12	70	70
	1 : 1·37 : 2·75	84	84

Although the first slip in the case of deformed as well as of plain bars generally takes place at about the same stress, the ultimate bond strength for deformed bars is relatively high, and the working grip stress may be taken as equal to that allowed for plain bars hooked at the ends, or from 25 to 50 per cent. greater than that for plain bars otherwise anchored.

WORKING STRESSES FOR STEEL.

The effective strength of a reinforced concrete member, assuming the integrity of the concrete, is governed by the

yield point rather than by the ultimate strength of the steel.

Therefore, the working stresses for steel should be based upon the yield point with due regard to the deformation of the metal. In members subject to flexure the elongation of the steel is necessarily proportionate to the stress, and as this is increased beyond a point corresponding with the ultimate tensile resistance of the concrete, the hair cracks distributed along the surface of the latter must be proportionately increased in width.

For this reason it is desirable that the working tensile stress in steel should be limited to about 20,000 lb. per square inch in ordinary construction.

Steel in Tension.—The working stresses generally adopted are taken at about 50 per cent. of the stress at the elastic limit or the yield point. Some regulations insist upon the use of mild steel and limit the working stress to 16,000 lb. per square inch.

There is no reason, however, why high-tension steel of approved quality should not be employed in many classes of construction at working stresses up to 20,000 lb. per square inch.

Steel in Compression.- The working compressive stresses in steel are usually specified at m times the compressive stress c in the surrounding concrete, or $15c$ when 15 is taken as the value of m, the modular ratio.

In designs where the compressive resistance of the concrete is neglected the stress in the steel may be taken at about 50 per cent. of the stress at the yield point, providing the stress is not such as to cause failure of the concrete by crushing. In designs of this class, the L.C.C. Regulations limit the working compressive stress to 16,000 lb. per square inch.

Steel in Shear.—As so-called shear reinforcement acts in resisting tensile stress and is rarely, if ever, called upon to withstand shearing stress, it is scarcely necessary to state values for the working shearing stress in steel.

Summary of Working Stresses for Steel.—Table XIII

gives the working stresses specified by the four authorities previously cited.

TABLE XIII.

WORKING STRESSES FOR STEEL.

Authority.	*Tension.*	*Compression.*	*Shear.*
	lb. per sq. in.	*lb. per sq. in.*	*lb. per sq. in.*
R.I.B.A. Committee .	16,000,* or 50 % stress at yield point	$15c$	12,000,* or $37\frac{1}{2}$ % stress at yield point
L.C.C. Regulations .	16,000*	$m \cdot c$ or 16,000* neglecting compressive resistance of concrete	—
American Committee .	16,000*	$15c$	—
French Commission .	50 % stress at elastic limit, or 40 % in members subject to alternating stresses	50 % stress at elastic limit, or 40 % in members subject to alternating stresses	—

WORKING STRESSES AND DATA

From the preceding paragraphs it will be seen that, unless fettered by hard and fast regulations, the designer has considerable scope for the exercise of judgment in the selection of appropriate working stresses in concrete and steel.

For this reason, the author abstains from the presentation of his own views in tabulated form. But as values of some kind must be employed for the various factors in the equations stated in this book, when numerical examples are being worked, Table XIV is added, giving working stresses and other data which are commonly adopted for qualities of concrete and steel in more or less general use.

* Values relating to mild steel with an ultimate strength of about 60,000 lb. per square inch.

TABLE XIV.

WORKING STRESSES AND OTHER DATA COMMONLY USED IN REINFORCED CONCRETE DESIGN.

Concrete: ultimate strength, 1,800 lb. per sq. in. at 28 days.
Steel: ultimate strength not less than 60,000 lb. per sq. in.
Modular Ratio: $m = 15$.

Material, Kind of Stress, etc.	*Working Stress or Value.*	*Remarks.*
CONCRETE:		
Tension . .	—	
Compression . .	600 lb. per sq. in.	Beams and compression members
Shear (simple) .	—	Web reinforcement designed to take total shear
Shear (as measure of diagonal tension)	—	
GRIP OR ADHESION:		
Plain bars, ends anchored but not hooked	60 lb. per sq. in.	
Plain bars (with hooked ends) and deformed bars	100 ,, ,,	
STEEL:		
Tension . .	16,000 ,, ,,	
Compression .	$15c$	c = compressive working stress for concrete
Compression (compressive strength of concrete neglected	16,000 lb. per sq. in.	

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